ETH Zürich FS 2016
D-MATH
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Brownian Motion and Stochastic Calculus
Solution 7

Solution 7-1

The left-hand side is indeed well defined because the process $X^t_\cdot := (X_{\cdot \wedge t})$ is in Λ_2. This process is progressively measurable and

$$E\left[\int_0^\infty X_{s\wedge t}^2 ds\right] = E\left[\int_0^t X_s^2 ds\right] \leq E\left[t\,(\sup_{0\leq s\leq t} X_s)^2\right].$$

By (2.48) of the lecture note, $\sup_{0\leq s\leq t} X_s$ has the same law as $|X_t|$, hence

$$E\left[\int_0^\infty X_{s\wedge t}^2 ds\right] \leq E[tX_t^2] < \infty.$$

Since both sides of the equation which needs to be proven are a.s. continuous, it is sufficient to prove that for any fixed $t \geq 0$

$$P - a.s. \int_0^t X_s \, dX_s = \frac{1}{2}\Big(X_t^2 - t\Big).$$

If we divide the interval $[0, t]$ into 2^n equal parts, we can approximate X^t by the simple process $X^{t,n} := \sum_{j=1}^{2^n} X_{(j-1)t/2^n} \mathbf{1}_{[(j-1)t/2^n, jt/2^n]}$ which is in Λ_1. Indeed, $X_n \to X$ in Λ_2 for the norm $L^2(dP \otimes ds)$ because

$$\begin{aligned}
E\left[\int_0^\infty (X_{s\wedge t} - X_{s\wedge t}^{t,n})^2 ds\right] &= E\left[\int_0^t \sum_{j=1}^{2^n} (X_s - X_{(j-1)t/2^n})^2 ds\right] \\
&= E\left[\sum_{j=1}^{2^n} \int_0^{2^{-n}} (X_{s+(j-1)t/2^n} - X_{(j-1)t/2^n})^2 ds\right] \\
&= 2^n E\left[\int_0^{2^{-n}} (X_s)^2 ds\right] \leq 2^n E\left[2^{-n}(\sup_{0\leq s\leq 2^{-n}} X_s)^2\right] \\
&= 2^n E[2^{-n} X_{2^{-n}}^2] = E[X_{2^{-n}}^2] \to 0.
\end{aligned}$$

We can now use the fact that the stochastic integral on Λ_2 is an isometry to deduce

$$\int_0^t X_s \, dX_s$$
$$= \lim_{n\to\infty} \int_0^t X_s^{t,n} dX_s$$
$$= \lim_{n\to\infty} \sum_{j=1}^{2^n} X_{(j-1)t/2^n}\left(X_{jt/2^n} - X_{(j-1)t/2^n}\right)$$
$$= \lim_{n\to\infty} \frac{1}{2}\sum_{j=1}^{2^n}\Big(\left(X_{jt/2^n} + X_{(j-1)t/2^n}\right)\left(X_{jt/2^n} - X_{(j-1)t/2^n}\right)\Big) - \lim_{n\to\infty}\frac{1}{2}\sum_{j=1}^{2^n}\left(X_{jt/2^n} - X_{(j-1)t/2^n}\right)^2$$
$$= \lim_{n\to\infty} \frac{1}{2}\sum_{j=1}^{2^n}\left(X_{jt/2^n}^2 - X_{(j-1)t/2^n}^2\right) - \lim_{n\to\infty}\frac{1}{2}\sum_{j=1}^{2^n}\left(X_{jt/2^n} - X_{(j-1)t/2^n}\right)^2$$
$$= \frac{1}{2}\left(X_t^2 - X_0^2\right) - \lim_{n\to\infty}\frac{1}{2}\sum_{j=1}^{2^n}\left(X_{jt/2^n} - X_{(j-1)t/2^n}\right)^2 \quad \text{a.s.}$$

Now, since $X_0 = 0$ and the quadratic variation of X at time t equals t a.s. (see Theorem 3.1 of the lecture notes), we get the desired result.

Solution 7-2

a) Let first assume that f is a step function, i.e.

$$f = \sum_{i=0}^{n-1} c_i \, 1_{[t_i, t_{i+1})}$$

where, $(c_i)_{i=0,\dots,n-1}$ are constants and $a = t_0 < t_1 < \dots < t_n = b$ is a partition of $[a,b]$. Since f is a simple process in Λ_1, by the definition in (4.37) of the lecture note, we have

$$\mathcal{J}_{a,b} = \int_a^b f(s)\, dB_s = \sum_{i=0}^{n-1} c_i \left(B_{t_{i+1}} - B_{t_i}\right)$$

As the increments of Brownian motion are independent and normally distributed, we get that $\mathcal{J}_{a,b}$ is normally distributed, too. Moreover, we get that

$$E\big[\mathcal{J}_{a,b}\big] = \sum_{i=0}^{n-1} c_i\, E\big[B_{t_{i+1}} - B_{t_i}\big] = 0$$

as well as

$$\operatorname{Var}\big(\mathcal{J}_{a,b}\big) = \sum_{i=0}^{n-1} c_i^2 \operatorname{Var}\big(B_{t_{i+1}} - B_{t_i}\big) = \sum_{i=0}^{n-1} c_i^2\left(t_{i+1} - t_i\right) = \int_a^b f^2(s)\, ds.$$

Now, let $f \in L^2([0,T])$. There exists a sequence of step functions $(f_n)_{n\in\mathbb{N}}$ such that $f_n \to f$ in $L^2([0,T])$ as $n \to \infty$. By the isometry property of the stochastic integral we get that

$$\int_a^b f_n(s)\, dB_s \longrightarrow \mathcal{J}_{a,b} = \int_a^b f(s)\, dB_s$$

in $L^2(\Omega, \mathcal{F}, P)$. Due to Series 2 Exercise 2, we get that $\mathcal{J}_{a,b}$ is normally distributed with expectation equal to 0 and, together with the above step,

$$\operatorname{Var}(\mathcal{J}_{a,b}) = \lim_{n\to\infty} \operatorname{Var}\Big(\int_a^b f_n(s)\, dB_s\Big) = \lim_{n\to\infty} \int_a^b f_n^2(s)\, ds = \int_a^b f^2(s)\, ds.$$

b) Let $(t_i)_{i\in I}$ be a finite partition of $[0,T]$ containing the points a, b, c, d. Assume first that $f = \sum_{i=0}^{n-1} c_i\, 1_{[t_i, t_{i+1})}$ is a step function. We then have

$$\mathcal{J}_{a,b} = \sum_{i\in I_1} c_i\, \big(B_{t_{i+1}} - B_{t_i}\big) \quad \text{and} \quad \mathcal{J}_{c,d} = \sum_{j\in I_2} c_j\, \big(B_{t_{j+1}} - B_{t_j}\big),$$

where $I_1 \cap I_2$ contains at most one point. Thus, as the increments of Brownian motion are independent and normally distributed, the random variables $\mathcal{J}_{a,b}$ and $\mathcal{J}_{c,d}$ are independent and normally distributed. Thus, we get that for any $\alpha, \beta \in \mathbb{R}$, $\alpha\mathcal{J}_{a,b} + \beta\mathcal{J}_{c,d}$ is a normally distributed random variable. Therefore, we conclude that the vector $(\mathcal{J}_{a,b}, \mathcal{J}_{c,d})$ is a Gaussian vector.
Now, let $f \in L^2([0,T])$. There exists a sequence of step functions $(f_n)_{n\in\mathbb{N}}$ such that $f_n \to f$ in $L^2([0,T])$ as $n \to \infty$. The same argument as in a) yields that for any $\alpha, \beta \in \mathbb{R}$, we have

$$\alpha\mathcal{J}_{a,b}^{(n)} + \beta\mathcal{J}_{c,d}^{(n)} \longrightarrow \alpha\mathcal{J}_{a,b} + \beta\mathcal{J}_{c,d}$$

in $L^2(\Omega, \mathcal{F}, P)$ (where the index (n) correspond to the function f_n). We deduce from the results above and Series 2 Exercise 2 that $\alpha\mathcal{J}_{a,b} + \beta\mathcal{J}_{c,d}$ is normally distributed and thus the vector $(\mathcal{J}_{a,b}, \mathcal{J}_{c,d})$ is a Gaussian vector. Moreover, using the identity

$$\operatorname{Cov}\big(\mathcal{J}_{a,b}, \mathcal{J}_{c,d}\big) = \frac{1}{2}\Big(\operatorname{Var}\big(\mathcal{J}_{a,b} + \mathcal{J}_{c,d}\big) - \operatorname{Var}\big(\mathcal{J}_{a,b}\big) - \operatorname{Var}\big(\mathcal{J}_{c,d}\big)\Big)$$

we deduce from the results above and Series 2 Exercise 2 that

$$\operatorname{Cov}\big(\mathcal{J}_{a,b}, \mathcal{J}_{c,d}\big) = \lim_{n\to\infty} \operatorname{Cov}\big(\mathcal{J}_{a,b}^{(n)}, \mathcal{J}_{c,d}^{(n)}\big) = 0.$$

Thus, we conclude that the random variables $\mathcal{J}_{a,b}$ and $\mathcal{J}_{c,d}$ are independent as $(\mathcal{J}_{a,b}, \mathcal{J}_{c,d})$ is a Gaussian vector.

Solution 7-3

a) The same method as in Exercise 2 b) yields that for any $0 \le t_1 < t_2 < ... < t_k \le T$, the vector $(\mathcal{J}_{0,t_1}, \mathcal{J}_{t_1,t_2}, ..., \mathcal{J}_{t_{k-1},t_k})$ is a Gaussian vector and all of its components are independent. For any constants $\alpha_1, ..., \alpha_k \in \mathbb{R}$, we have the (algebraic) identity

$$\alpha_1 \mathcal{J}_{t_1} + ... + \alpha_k \mathcal{J}_{t_k} = (\alpha_1 + ... + \alpha_k)\mathcal{J}_{0,t_1} + (\alpha_2 + ... + \alpha_k)\mathcal{J}_{t_1,t_2} + ... + \alpha_k \mathcal{J}_{t_{k-1},t_k}.$$

Thus, the random variable $\alpha_1 \mathcal{J}_{t_1} + ... + \alpha_k \mathcal{J}_{t_k}$ is normally distributed being a sum of independent normally distributed random variables. Since $0 \le t_1 < t_2 < ... < t_k \le T$ was arbitrarily chosen, we conclude that the process $\mathcal{J} := (\mathcal{J}_t)_{t \in [0,T]}$ is a Gaussian process.
We now calculate its Covariance function Γ : let $s < t$. By the previous exercise, we know that $\mathcal{J}_{0,s}$ and $\mathcal{J}_{s,t}$ are independent and have expectations equal to 0. Hence $\mathcal{J}$ is a centered process. Moreover, always by the previous exercise,

$$\begin{aligned}\Gamma(s,t) &= E\big[\mathcal{J}_s\,\mathcal{J}_t\big] - E\big[\mathcal{J}_s\big]\,E\big[\mathcal{J}_t\big] = E\big[\mathcal{J}_s\,\mathcal{J}_t\big] = E\big[\mathcal{J}_s(\mathcal{J}_{0,s} + \mathcal{J}_{s,t})\big] \\ &= E\big[\mathcal{J}_{0,s}^2\big] = \mathrm{Var}\big(\mathcal{J}_{0,s}\big) = \int_0^s f^2(u)\,du.\end{aligned}$$

Thus, for any s, t we obtain that

$$\Gamma(s,t) = \int_0^{\min(s,t)} f^2(u)\,du.$$

b) By the previous exercise, we know that for any $t \ge 0$, $E[\mathcal{J}_t] = 0$. For $g(t) := \int_0^t f^2(s)\,ds$, it is clear that the process $Y := (Y_t)_{t \in [0,T]}$ defined by

$$Y_t := B_{g(t)}$$

is a Gaussian process with $E[Y_t] = 0$ for any $t \ge 0$ as Brownian motion is. Moreover, for any $s, t \in [0,T]$, due to the Covariance property of Brownian motion, we obtain that

$$\mathrm{Cov}\big(B_{g(s),g(t)}\big) = \min\big(g(s), g(t)\big) = \Gamma(s,t).$$

We conclude that the processes $\mathcal{J}$ and Y are both Gaussian processes with same expectations and Covariance functions, thus have the same finite dimensional marginal distributions.