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Brownian Motion and Stochastic Calculus
Solution 8

Solution 8-1

a) We first show that for any $t \geq 0$, $M_{\tau \wedge t}$ is integrable. We argue in the same way as in Series 6 Exercise 1, we repeat here its argument again. Fix any $t \geq 0$. Then, $\tau \wedge t$ is a bounded stopping time. Consider the sequence of stopping times $(\tau_n)_{n \in \mathbb{N}}$ defined by

$$\tau_n(\omega) := \sum_{k=1}^{\infty} \frac{k}{2^n} \mathbf{1}_{\{\frac{k-1}{2^n} \leq \tau(\omega) \wedge t < \frac{k}{2^n}\}}$$

The sequence $(\tau_n)_{n \in \mathbb{N}}$ decreases and converges to $\tau \wedge t$ when we let n goes to infinity. By the discrete stopping theorem, $(M_{\tau_n})_{n \in \mathbb{N}}$ is a backward martingale with respect to $(\mathcal{G}_{\tau_n})_{n \in \mathbb{N}}$. We conclude from Series 5 exercise 3 b) that the sequence of random variables $(M_{\tau_n})_{n \in \mathbb{N}}$ is uniformly integrable. Now, by right-continuity of M we get that $\lim_{n \to \infty} M_{\tau_n} = M_{\tau \wedge t}$. Therefore, due to the uniform integrability, we get that also $\lim_{n \to \infty} E[|M_{\tau_n}|] = E[|M_{\tau \wedge t}|]$. Moreover, applying Fatou, we obtain for any $M > 0$ that

$$E\big[|M_{\tau \wedge t}|\big] \leq \liminf_{n \to \infty} E\big[|M_{\tau_n}|\big] \leq M + \sup_{n \in \mathbb{N}} E\big[|M_{\tau_n}| \, \mathbf{1}_{\{|M_{\tau_n}| \geq M\}}\big]$$

which is finite for big enough M by the uniform integrability of $(M_{\tau_n})_{n \in \mathbb{N}}$. So we get the integrability of $M_{\tau \wedge t}$. Now, fix any $0 \leq s \leq t < \infty$. We need to show that

$$E\big[M_{\tau \wedge t} \,\big|\, \mathcal{G}_s\big] = M_{\tau \wedge s} \quad \text{a.s.}$$

Observe that $M_{\tau \wedge t} \mathbf{1}_{\{\tau \leq s\}} = M_{\tau \wedge s} \mathbf{1}_{\{\tau \leq s\}}$ and is $\mathcal{G}_s$-measurable. We deduce from Series 6 Exercise 1 that

$$\begin{aligned} E\big[M_{\tau \wedge t} \,\big|\, \mathcal{G}_s\big] &= E\big[M_{\tau \wedge t} \, \mathbf{1}_{\{\tau \leq s\}} + M_{\tau \wedge t} \mathbf{1}_{\{\tau > s\}} \,\big|\, \mathcal{G}_s\big] \\ &= M_{\tau \wedge s} \mathbf{1}_{\{\tau \leq s\}} + E\big[M_{\tau \wedge t} \,\big|\, \mathcal{G}_s\big] \; \mathbf{1}_{\{\tau > s\}} \\ &= M_{\tau \wedge s} \mathbf{1}_{\{\tau \leq s\}} + M_s \mathbf{1}_{\{\tau > s\}} \\ &= M_{\tau \wedge s} \quad \text{a.s.} \end{aligned}$$

b) As M is a continuous local martingale, there exists by definition a sequence of stopping times $(T_n)_{n \in \mathbb{N}}$, P-a.s. tending to infinity, such that for each n, the stopped process $M^{T_n} := (M_{T_n \wedge t})_{t \geq 0}$ is a continuous martingale. Consider the sequence of stopping times $(\tau_n)_{n \in \mathbb{N}}$ defined by

$$\tau_n := \inf\big\{t \geq 0 \,\big|\, |M_t| > n\big\}$$

As M_0 is bounded, we get that P-a.s. τ_n tends to infinity. Moreover, by (left-) continuity of M, for any n, the process $M^{\tau_n} := (M_{\tau_n \wedge t})_{t \geq 0}$ is uniformly bounded by n. Define the sequence of stopping times $(S_n)_{n \in \mathbb{N}}$ by $S_n := T_n \wedge \tau_n$. By construction, we get that P-a.s. S_n tends to infinity and $M^{S_n} := (M_{S_n \wedge t})_{t \geq 0}$ is uniformly bounded by n. Moreover, due to a), we get that for any n, $M^{S_n} = (M^{T_n})^{\tau_n}$ is a martingale.

Solution 8-2

a) Without loss of generality, suppose $(X_t)_{t\geq 0}$ is a local martingale with $X_0 = 0$ and let B be a constant such that $|X_t| \leq B$ for all $t \geq 0$. Let $(\tau_k)_{k\in\mathbb{N}}$ be a localizing sequence for X, i.e. it is a non decreasing sequence of stopping times such that $(X_{t\wedge\tau_k})_{t\geq 0}$ is a martingale for any k and $\tau_k \nearrow +\infty$ a.s. Fix $s \leq t$, by the martingale property we have

$$E\big[X_{t\wedge\tau_k} \,\big|\, \mathcal{G}_s\big] = X_{s\wedge\tau_k} \quad \text{a.s.}$$

By the dominated convergence theorem, which we can apply by the uniform boundedness of X, we get that

$$E\big[X_t \,\big|\, \mathcal{G}_s\big] = \lim_{k\to\infty} E\big[X_{t\wedge\tau_k} \,\big|\, \mathcal{G}_s\big] = \lim_{k\to\infty} X_{s\wedge\tau_k} = X_s \;\; \text{a.s.}$$

b) Let $(\tau_k)_{k\in\mathbb{N}}$ be a localizing sequence for X (see **a)** for the definition). Then, applying the local martingale property, we have for any $0 \leq s \leq t \leq T$ that

$$X_{s\wedge\tau_k} = E[X_{t\wedge\tau_k}|\mathcal{G}_s] \quad \text{a.s.}$$

Since X is nonnegative, we can apply Fatou's lemma to get for any $0 \leq s \leq t \leq T$ that

$$X_s = \lim_{k\to\infty} X_{s\wedge\tau_k} = \liminf_{k\to\infty} E\big[X_{t\wedge\tau_k} \,\big|\, \mathcal{G}_s\big] \geq E\big[\liminf_{k\to\infty} X_{t\wedge\tau_k} \,\big|\, \mathcal{G}_s\big] = E\big[X_t|\mathcal{G}_s\big] \quad \text{a.s.} \qquad (1)$$

Moreover as X is nonnegative, we obtain by applying Fatou's Lemma that for any $t \in [0, T]$

$$E\big[X_t\big] = E\big[\liminf_{k\to\infty} X_{t\wedge\tau_k}\big] \leq \liminf_{k\to\infty} E\big[X_{t\wedge\tau_k}\big] = E\big[X_0\big] < \infty$$

and so $(X_t)_{t\geq 0}$ is a supermartingale.
Now, take the expectation on both sides in (1), we get

$$E[X_s] \geq E[X_t]$$

for all $0 \leq s \leq t \leq T$. In particular, using monotonicity of the expectation for a supermartingale, we have

$$E[X_0] \geq E[X_s] \geq E[X_t] \geq E[X_T] \quad \text{for all } 0 \leq s \leq t \leq T. \qquad (2)$$

Using the assumption $E[X_T] = E[X_0]$, we see that the previous inequalities in (2) are all equalities. If the inequality in (1) was strict on a set of positive probability, we would have $E[X_s] > E[X_t]$, which gives a contradiction, and so the equality must hold with probability one. Thus, X is a martingale.

Solution 8-3

a) The continuous processes

$$(X_t)_{t\geq 0} := \left(\frac{B_t + \widetilde{B}_t}{\sqrt{2}}\right)_{t\geq 0} \quad \text{and } (Y_t)_{t\geq 0} := \left(\frac{B_t - \widetilde{B}_t}{\sqrt{2}}\right)_{t\geq 0},$$

are, by independence of B and $\tilde{B}$ and as both B and $\tilde{B}$ are Gaussian processes, themselves Gaussian processes with the same expectation and covariance as Brownian motion. Therefore, they are both Brownian motions. Moreover, by the polarization formula,

$$\begin{aligned}
4\langle B, \widetilde{B}\rangle_t &= \langle B+\widetilde{B}\rangle_t - \langle B-\widetilde{B}\rangle_t \\
&= 2\left\langle \frac{B+\widetilde{B}}{\sqrt{2}} \right\rangle_t - 2\left\langle \frac{B-\widetilde{B}}{\sqrt{2}} \right\rangle_t \\
&= 2\langle X\rangle_t + 2\langle Y\rangle_t = 2t - 2t = 0.
\end{aligned}$$

b) By independence of B and $\tilde{B}$ and as both B and $\tilde{B}$ are Gaussian processes, we get directly that W is a Gaussian process. Moreover, W is continuous and has the same expectation and covariance as Brownian motion, therefore it is itself a Brownian motion. Moreover, we have

$$\langle W, B\rangle_t = \langle \rho B, B\rangle_t + \langle \sqrt{1-\rho^2}\widetilde{B}, B\rangle_t = \rho\langle B\rangle_t = \rho t.$$

So W and B are two correlated Brownian motions with correlation ρ.