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Brownian Motion and Stochastic Calculus
Solution 9

Solution 9-1

Let $(S_n)_{n\in\mathbb{N}}$ be a sequence of stopping times, P-a.s. tending to infinity, such that for each n, the stopped process $M^{S_n} := (M_{S_n\wedge t})_{t\geq 0}$ is a square integrable continuous martingale. Moreover, let $(T_n)_{n\in\mathbb{N}}$ be a sequence of stopping times, P-a.s. tending to infinity, such that for each n, the stopped process $(M^2 - \langle M\rangle)^{T_n}$ is a square integrable continuous martingale. Set for each $n \in \mathbb{N}$, $\tau_n := S_n \wedge T_n$. By construction, $(\tau_n)_{n\in\mathbb{N}}$ is a sequence of stopping times, P-a.s. tending to infinity, such that for each n, both process M^{τ_n} and $(M^2 - \langle M\rangle)^{\tau_n}$ are square integrable continuous martingales. We then have for each $t \geq 0$ that

$$E[M^2_{\tau_n\wedge t}] = E[\langle M\rangle_{\tau_n\wedge t}]. \tag{1}$$

By monotonicity, we obtain that for each $t \geq 0$

$$\sup_{n\in\mathbb{N}} E[M^2_{\tau_n\wedge t}] = \sup_{n\in\mathbb{N}} E[\langle M\rangle_{\tau_n\wedge t}] \leq E[\langle M\rangle_t] =: K(t) < \infty. \tag{2}$$

Recall that a family $\{X_a\}_{a\in A}$ is uniformly integrable if and only if there exists a non-negative increasing convex function $G(t)$ such that $\lim_{t\to\infty} G(t)/t = \infty$ and $\sup_\alpha E(G(|X_a|)) < \infty$. (Also called the de la Vallée-Poussin Lemma.) Therefore, $(M_{\tau_n\wedge t})_{n\in\mathbb{N}}$ is bounded in L^2 and hence is uniformly integrable.

We can thus pass to the limit in the equality

$$E[M_{\tau_n\wedge t}|\mathcal{F}_s] = M_{\tau_n\wedge s} \quad \forall\, n \in \mathbb{N},$$

obtaining that $E[M_t|\mathcal{F}_s] = M_s$, so that M is a martingale. Moreover, by applying Fatou's lemma to (2) we obtain that for each $t \geq 0$

$$E[M_t^2] \leq \liminf_{n\to\infty} E[M^2_{\tau_n\wedge t}] \leq K(t) < \infty,$$

which finally yields that M is a square integrable martingale.

Solution 9-2

a) As we want to apply Itô's formula to $F'(X)$ we take $F \in C^3$. With Itô's formula applied to $F'(X)$ we get $\langle F'(X), X\rangle_t = \int_0^t F''(X_s)d\langle X\rangle_s$. Thus

$$\int_0^t F'(X_s) \circ dX_s = \int_0^t F'(X_s)\, dX_s + \frac{1}{2}\langle F'(X), X\rangle_t = \int_0^t F'(X_s)\, dX_s + \frac{1}{2}\int_0^t F''(X_s)d\langle X\rangle_s.$$

This equals $F(X_t) - F(X_0)$ by Itô's formula.

For the second part, take $X = Y = W$ to be a standard Brownian motion. Then $\int_0^t Y_s \circ dX_s = \int_0^t W_s\, dW_s + t/2$. If the Stratonovich integral were a local martingale, then $t/2$ would also be a local martingale, which is obviously not true.

b) We can find a sequence of partitions $(\Pi_n)_{n\in\mathbb{N}}$ such that

$$\langle Y, X\rangle_t = \lim_{n\to\infty} \sum_{t_i\in\Pi_n} (Y_{t_{i+1}\wedge t} - Y_{t_i\wedge t})\,(X_{t_{i+1}\wedge t} - X_{t_i\wedge t})$$

simultaneously for all $t \geq 0$ (see notes: using integration by parts formula and polarization formula that defines $\langle Y, X\rangle_t$) as well as

$$\int_0^t Y_s\, dX_s = \lim_{n\to\infty} \sum_{t_i\in\Pi_n} (Y_{t_i\wedge t})\,(X_{t_{i+1}\wedge t} - X_{t_i\wedge t})$$

simultaneously for all $t \geq 0$. Therefore, we get the result directly from the definition of the Stratonovich integral.

Solution 9-3

a) Since the 3-dimensional Brownian motion $B = (B^1, B^2, B^3)$ takes values in the open set $D := \mathbb{R}^d \setminus \{-x\}$ P-a.s., we can apply Itô's formula to $M_t = f(B_t)$ with $f : D \to (0,\infty)$ given by $f(y) := \frac{1}{|x+y|}$.

For $i = 1, 2, 3$, we have

$$\frac{\partial f}{\partial y^i}(y) = -\frac{x^i + y^i}{|x+y|^3}, \quad \frac{\partial^2 f}{(\partial y^i)^2}(y) = \frac{-|x+y|^2 + 3(x^i+y^i)^2}{|x+y|^5}.$$

It follows that $\Delta f = \frac{\partial^2 f}{(\partial y^1)^2} + \frac{\partial^2 f}{(\partial y^2)^2} + \frac{\partial^2 f}{(\partial y^3)^2} = 0$ on D. Hence, Itô's formula yields

$$M_t = M_0 + \int_0^t \nabla f(B_s)\, dB_s + \frac{1}{2}\int_0^t \Delta f(B_s)\, ds = \frac{1}{|x|} - \sum_{i=1}^3 \int_0^t \frac{x^i + B_s^i}{|x+B_s|^3}\, dB_s^i.$$

Thus, M is a continuous local martingale.
Let's show the second part. For $t > 0$,

$$\begin{aligned}
E\left[|M_t|^2, |M_t| \geq \frac{2}{|x|}\right] &= (2\pi t)^{-\frac{3}{2}} \int_{|x+y|\leq \frac{|x|}{2}} \frac{1}{|x+y|^2} \exp\left(-\frac{|y|^2}{2t}\right) dy \\
&= (2\pi t)^{-\frac{3}{2}} \int_{|y|\leq \frac{|x|}{2}} \frac{1}{|y|^2} \exp\left(-\frac{|y-x|^2}{2t}\right) dy \\
&\leq (2\pi t)^{-\frac{3}{2}} \int_{|y|\leq \frac{|x|}{2}} \frac{1}{|y|^2} \exp\left(-\frac{(|x|-|y|)^2}{2t}\right) dy \\
&\leq (2\pi t)^{-\frac{3}{2}} \exp\left(-\frac{|x|^2}{8t}\right) \int_{|y|\leq \frac{|x|}{2}} \frac{1}{|y|^2}\, dy.
\end{aligned}$$

The integral term in the preceding expression is finite since the domain of integration is 3-dimensional. Moreover, the function $t \mapsto (2\pi t)^{-\frac{3}{2}} \exp\left(-\frac{|x|^2}{8t}\right)$ is continuous on $(0,\infty)$ and converges to 0 as $t \to 0$ and $t \to \infty$, hence it is bounded on $(0,\infty)$. It follows that M is bounded in L^2.

b) For $t > 0$, using spherical coordinates,

$$\begin{aligned}
E[M_t] &= (2\pi t)^{-3/2} \int_{\mathbb{R}^3} \frac{1}{|x+y|} \exp\left(-\frac{|y|^2}{2t}\right) dy \\
&= (2\pi t)^{-3/2} \int_{\mathbb{R}^3} \frac{1}{|y|} \exp\left(-\frac{|y-x|^2}{2t}\right) dy \\
&\leq (2\pi t)^{-3/2} \int_{\mathbb{R}^3} \frac{1}{|y|} \exp\left(-\frac{(|y|-|x|)^2}{2t}\right) dy \\
&= (2\pi t)^{-3/2} \int_0^\infty \int_0^{2\pi} \int_0^\pi \frac{1}{r} \exp\left(-\frac{(r-|x|)^2}{2t}\right) r^2 \sin\varphi \, d\varphi d\theta dr \\
&= 4\pi (2\pi t)^{-3/2} \int_0^\infty r \exp\left(-\frac{(r-|x|)^2}{2t}\right) dr \\
&= 4\pi (2\pi t)^{-3/2} \int_{-|x|}^\infty (r+|x|) \exp\left(-\frac{r^2}{2t}\right) dr \\
&= 4\pi (2\pi t)^{-3/2} \left(\int_{-|x|}^\infty r \exp\left(-\frac{r^2}{2t}\right) dr + |x| \int_{-|x|}^\infty \exp\left(-\frac{r^2}{2t}\right) dr \right) \\
&\leq 4\pi (2\pi t)^{-3/2} \left(\left[-t \exp\left(-\frac{r^2}{2t}\right) \right]_{-|x|}^\infty + |x| \sqrt{2\pi t} \right) \\
&= 4\pi (2\pi t)^{-3/2} \left(t \exp\left(-\frac{|x|^2}{2t}\right) + |x| \sqrt{2\pi t} \right) = O\left(t^{-\frac{1}{2}}\right) \quad (t \to \infty).
\end{aligned}$$

Hence, $E[M_t] \to 0$ as $t \to \infty$. Since $E[M_0] = \frac{1}{|x|} > 0$, M cannot be a martingale.