Exercise sheet 1

1. Let V be a finite dimensional real vector space and let $\Lambda \subseteq V$ be a subgroup. Prove that the following statements are equivalent:

1. $\Lambda \subseteq \mathbb{R}^n$ is discrete and V/Λ is compact;
2. $\Lambda \subseteq \mathbb{R}^n$ is discrete and Λ is free with rank $\dim_{\mathbb{R}}(V)$;
3. Λ is generated by a basis of V.

We call such a subgroup of V a *lattice*. For $g \in \mathbb{Z}_{>0}$ we say that $\Lambda \subseteq \mathbb{C}^g$ is a lattice if it is mapped to a lattice of $\mathbb{R}^{2g}$ by the isomorphism $\mathbb{C}^g \cong \mathbb{R}^{2g}$ induced by real and imaginary parts on each component.

2. Let f be a non-constant meromorphic function on $\mathbb{C}$. We say that $\lambda \in \mathbb{C}$ is a *period* of f if $f(z) = f(z + \lambda)$ for all $z \in \mathbb{C}$. Let Λ be the set of periods of f. Prove that $\Lambda \subseteq \mathbb{C}$ is a discrete subgroup. Which ranks can Λ have? [You may use Theorem 2.2 from Debarre's book].

3. Let $\Lambda \subseteq \mathbb{R}^n$ be a lattice. Find all continuous group homomorphisms

$$\mathbb{R}^n/\Lambda \longrightarrow \mathbb{R}^n/\Lambda.$$

Which of those group homomorphisms are isomorphisms?

[*Hint:* What are the continuous group homomorphisms $\mathbb{R} \longrightarrow \mathbb{R}$?]

4. Let $\Lambda \subseteq \mathbb{C}$ be a lattice generated by 1 and $\tau \in \mathbb{C}$. Find all $\mathbb{C}$-linear maps $\phi : \mathbb{C} \longrightarrow \mathbb{C}$ such that $\phi(\Lambda) \subseteq \Lambda$, for

- $\tau = 2\pi i$;
- $\tau = e^{\frac{\pi i}{3}}$;
- $\tau = i + \sqrt{7}$.

5. Consider a lattice $\Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2 \subseteq \mathbb{C}$ and an entire function $\vartheta : \mathbb{C} \longrightarrow \mathbb{C}$ such that

$$\vartheta(z) = a_1\vartheta(z + \omega_1), \ \vartheta(z) = a_2\vartheta(z + \omega_2), \quad \forall z \in \mathbb{C},$$

for some $a_1, a_2 \in \mathbb{C}$. Prove that there exist $b, c \in \mathbb{C}$ such that

$$\vartheta(z) = be^{cz}.$$