Exercise Sheet 10

1. Let $X_\tau = \mathbb{C}^g/\Lambda_\tau$ be an abelian variety with $\Lambda_\tau = \tau\mathbb{Z}^g \oplus \Delta\mathbb{Z}^g$, where $\tau \in \mathcal{H}_g$ and $\Delta = \operatorname{diag}(d_1, \dots, d_g)$ has integer entries $d_j > 0$ such that $d_j | d_{j+1}$. For each $a, b \in \mathbb{R}^g$, we consider the Riemann theta function $\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (\cdot, \tau) : \mathbb{C}^g \longrightarrow \mathbb{C}$ defined by

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (z, \tau) := \sum_{m \in \mathbb{Z}^g} \exp i\pi ({}^t(m+a)\tau(m+a) + 2\,{}^t(m+a)(z+b)).$$

a) Check that $\vartheta \begin{bmatrix} a \\ 0 \end{bmatrix} (\cdot, \tau)$ is a theta function for Λ_τ for $a \in \Delta^{-1}\mathbb{Z}^g$.

b) Express the automorphy factor of $\vartheta \begin{bmatrix} a \\ 0 \end{bmatrix} (\cdot, \tau)$ in terms of the following data:

$$\begin{aligned} \omega(\tau p + q, \tau p' + q') &= {}^t p' q - {}^t q' p, \\ H(z, z') &= {}^t \bar{z} (\operatorname{Im}\tau) z, \\ (H - B)(\tau p + \Delta q) &= -2i\,{}^t p z, \\ \alpha(\tau p + \Delta q) &= (-1)^{{}^t p \Delta q}, \\ \ell &= 0 \end{aligned}$$

Deduce that $\vartheta \begin{bmatrix} a \\ 0 \end{bmatrix} (\cdot, \tau)$ is associated with the line bundle $L_\tau := L(H, \alpha)$ over X_τ [*Hint*: See Remark 6.5 from Debarre's book]

c) Show that if $d_1 \geq 2$, then for every z and τ there exists $a \in \Delta^{-1}\mathbb{Z}^g$ such that $\vartheta \begin{bmatrix} a \\ 0 \end{bmatrix} (z, \tau) \neq 0$. [*Hint:* Use Lefschetz's Theorem]

Let m be an integer dividing d_1. Define $\alpha_m(\tau p + \Delta q) := (-1)^{\frac{1}{m}\,{}^t p \Delta q}$ for $p, q \in \mathbb{Z}^g$.

d) Show that the pair $(\frac{1}{m}H, \alpha_m)$ is the type of a line bundle M_τ over X_τ such that $M_\tau^{\otimes m} = L_\tau$.

e) Prove that the $z \mapsto \vartheta \begin{bmatrix} a \\ 0 \end{bmatrix} \left(\frac{z}{m}, \frac{\tau}{m}\right)$ are theta functions associated with M_τ for $a \in m\Delta^{-1}\mathbb{Z}^g/\mathbb{Z}^g$.

Please turn over!

2. Let r be an integer. Find an invertible matrix with coefficients independent of z and τ which transforms the r^{2g} functions

$$\vartheta\left[\begin{array}{c} a \\ 0 \end{array}\right](rz,\tau),\ a \in \frac{1}{r^2}\mathbb{Z}^g/\mathbb{Z}^g$$

into the r^{2g} functions

$$\vartheta\left[\begin{array}{c} a \\ b \end{array}\right](r^2z,r^2\tau),\ a,b \in \frac{1}{r^2}\mathbb{Z}^g/\mathbb{Z}^g.$$

3. Let G be a finite subgroup of $\mathrm{GL}_n(\mathbb{C})$.

a) Find an example in which G is not generated by pseudoreflections, and show that $G\backslash\mathbb{C}^n$ is singular.

b) Find an interesting example in which G is generated by pseudoreflections, and show that $G\backslash\mathbb{C}^n$ is not singular.

c) What happens if $n = 1$?