Exercise sheet 2

1. Let ω be a closed 2-form on a complex torus $X = V/\Lambda$. For a fixed basis $\{x_j\}$ of Λ, write

$$\omega(x) = \sum_{j<k} \omega_{jk}(x) dx_j \wedge dx_k,$$

where the ω_{jk} are Λ-periodic functions, so that they can be expanded into Fourier series with respect to $\{x_j\}$:

$$\omega_{jk}(x) = \sum_{m \in \mathbb{Z}^g} \omega_{jk}^{(m)} e^{2i\pi {}^t m x}, \quad \omega_{jk}^{(m)} \in \mathbb{C}.$$

(a) Set, for each $m \in \mathbb{Z}^g$,

$$\omega^{(m)} := \sum_{j<k} \omega_{jk}^{(m)} e^{2i\pi {}^t m x} dx_j \wedge dx_k.$$

Show that $\omega^{(m)}$ is a closed 2-form on X.

(b) Show that $\omega^{(m)}$ is an exact 2-form on X for any $m \neq 0$. Conclude that ω is cohomologous to a constant form.

2. Let $n \in \mathbb{Z}_{>0}$ and consider the canonical projection $p : \mathbb{C}^{n+1} \setminus \{0\} \longrightarrow \mathbb{P}^n$. For $j = 0, \ldots, n$, let $U_j = \{[x_0 : \cdots : x_n] \in \mathbb{P}^n | x_j \neq 0\}$, and consider the standard charts

$$\phi_j : \mathbb{C}^n \xrightarrow{\iota_j} \{(x_0, \ldots, x_n) \in \mathbb{C}^{n+1} : x_j = 1\} =: \tilde{U}_j \xrightarrow{p} U_j$$
$$(x_0, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) \mapsto (x_0, \ldots, x_{j-1}, 1, x_{j+1}, \ldots, x_n)$$

of $\mathbb{P}^n$.

(a) Let ω be a complex r-form on $\mathbb{P}^n$. Show that $p^*\omega(z)(v_1, \ldots, v_r) = 0$ whenever either v_1 and z, or v_1 and $\bar{z}$ are colinear.

(b) Conversely, suppose that a complex r-form $\tilde{\omega}$ on $\mathbb{C}^{n+1} \setminus \{0\}$ is invariant under the action of $\mathbb{C}^\times$ and has in addition the property in (a). Show that the differential forms $\iota_j^* \tilde{\omega}$ satisfy the appropriate compatibility conditions, and hence define a differential form on $\mathbb{P}^n$ with $p^*(\omega) = \tilde{\omega}$.

Please turn over!

3. Consider a positive integer n and write $||z||^2 = \sum_{j=0}^n z_j \bar{z}_j$. Then

$$\tilde{\omega} = \frac{i}{2\pi} \frac{||z||^2 \left(\sum_{j=0}^n dz_j \wedge d\bar{z}_j\right) - \left(\sum_{j=0}^n \bar{z}_j dz_j\right) \wedge \left(\sum_{k=0}^n z_k d\bar{z}_k\right)}{||z||^4}$$

defines a real, type (1,1) differential 2-form on $\mathbb{P}^n$, called the Fubini-Study form ω_{FS}.

(a) Check that indeed $\tilde{\omega}$ meets the necessary conditions from Exercise 2.

(b) Check that $\omega_{\mathrm{FS}} = \frac{i}{2\pi}\partial\bar{\partial}\log||z||^2$.[i]

(c) Prove that ω_{FS} is closed and invariant under the action of the unitary group $U(n+1)$.

4. Prove that a countable intersection of open dense subsets of $\mathbb{R}^N$ is dense.

5. Identify $\mathbb{C}^g$ with $\mathbb{R}^{2g}$. To each invertible real matrix M, one associates the lattice Γ_M in $\mathbb{C}^g$ generated by the columns of M. One thus parametrizes the set of lattices in $\mathbb{C}^g$ by the open dense set (in the real Zariski topology) $U = \mathrm{GL}_{2g}(\mathbb{R}) \subseteq \mathbb{R}^{4g^2}$.

(a) Show that there is a countable family $(Z_n)_{n\in\mathbb{Z}_{>0}}$ of real algebraic hypersurfaces in $\mathbb{R}^{4g^2}$ such that, for any matrix M in $U \setminus \bigcup_{n\in\mathbb{Z}_{>0}} Z_n$, the 2×2 minors in M^{-1} are linearly independent over $\mathbb{Z}$.

(b) Deduce: when $g \geq 2$ and $M \in U \setminus \bigcup_{n\in\mathbb{Z}_{>0}} Z_n$, the only bilinear, type $(1,1)$, alternating $\mathbb{R}$-form on $\mathbb{C}^g$ which is integral on Γ_M is the trivial one.

(c) Conclude that a "very general" complex torus (in a sense to be made precise) of dimension ≥ 2 cannot be holomorphically embedded in projective space. (*Hint:* You might use Exercise 4).

[i] Given a differential form ω of type (p,q) over a complex manifold, it can be proved that it decomposes uniquely as a sum $d\omega = \partial\omega + \bar{\partial}\omega$, where $\partial\omega$ is of type $(p+1,q)$ while $\bar{\partial}\omega$ is of type $(p,q+1)$. This allows to define differential operators ∂ and $\bar{\partial}$. From $d^2=0$, one easily deduces that $\partial^2 = \bar{\partial}^2 = \partial\bar{\partial} + \bar{\partial}\partial = 0$.