Exercise Sheet 3

1. Let $V \cong \mathbb{C}^g$ be a complex vector space and let $\Lambda \subseteq V$ be a lattice. Prove that for an entire function $f : V \longrightarrow \mathbb{C}$ the following conditions are equivalent:

i) f is a nowhere vanishing theta function on V associated to Λ;

ii) $f(z) = e^{Q(z)}$, where $Q(z)$ is a polynomial of degree ≤ 2.

2. Consider a lattice $\Lambda = \mathbb{Z} \oplus \tau\mathbb{Z} \subseteq \mathbb{C}$.

a) Prove that an entire function $f : \mathbb{C} \longrightarrow \mathbb{C}$ is a theta function associated to Λ if and only if $(f'/f)'$ is Λ-periodic.

Consider the *Weierstrass σ-function*

$$\sigma(z) := z \prod_{\lambda \in \Lambda \setminus \{0\}} \left(1 - \frac{z}{\lambda}\right) e^{\frac{z}{\lambda} + \frac{z^2}{2\lambda^2}}.$$

b) Prove that σ is a theta function associated to Λ. Find its type.

c) Find the divisor of σ.

Consider the Riemann theta functions with respect to Λ, defined for $a, b \in \mathbb{R}$ as

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (z) := \sum_{m \in \mathbb{Z}} e^{2\pi i \left(\frac{1}{2}\tau(m+a)^2 + (m+a)(z+b)\right)}.$$

d) For $a, b, a', b' \in \mathbb{R}$, express the divisor of $\vartheta \begin{bmatrix} a' \\ b' \end{bmatrix}$, considered as a subset of $\mathbb{C}$, as a translate of the divisor of $\vartheta \begin{bmatrix} a \\ b \end{bmatrix}$.

e) Find the divisor of $\vartheta \begin{bmatrix} a \\ b \end{bmatrix}$ for each $a, b \in \mathbb{R}$.

f) Show that there exist constants $u, v \in \mathbb{C}$ such that, for any $z \in \mathbb{C}$,

$$\sigma(z) = e^{uz^2 + v} \vartheta \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} (z).$$