Exercise Sheet 4

1. Show that for a "very general" complex torus $X = V/\Lambda$ (see Exercise 5 from Exercise Sheet 2) of dimension at least 2, any theta function associated to Λ is trivial and any meromorphic function on X is constant.

2. Let W be a complex vector space of dimension n. For any integer $d \leq n$, consider the set

$$G(d, W) := \{V \leq W \,|\, \dim_{\mathbb{C}}(V) = d\}.$$

Notice that $G(1, W) = \mathbb{P}W$.

a) Explain how $G(d, W)$ can be seen as

(i) a topological space.

(ii) a complex manifold (*Hint:* for every $U \in G(n-d, W)$ we can identify the subset $\{\Pi \in G(d, W) \,|\, \Pi \cap U = 0\}$ of $G(d, W)$ with $\mathrm{Hom}(W/U, U)$).

(iii) a projective algebraic variety (*Hint:* it could help to solve the rest of the exercise beforehand).

The set $G(d, W)$ is known as a *Grassmannian manifold.* What is its dimension?

b) Construct a line bundle $L \longrightarrow G(d, W)$, for which the fiber over a point $\Pi \in G(d, W)$ is the line $\bigwedge^d \Pi$. We denote this line bundle by $\mathcal{O}_{G(d,W)}(-1)$.

c) Prove that the map $u : G(d, W) \longrightarrow \mathbb{P}(\bigwedge^d W)$ sending $\Pi \mapsto \bigwedge^d \Pi$ is holomorphic and injective, and that $u^*\mathcal{O}_{\mathbb{P}(\bigwedge^d W)}(1) = \mathcal{O}_{G(d,W)}(1)$.

3. Let W be a complex vector space and let m be an integer. Show that the vector space of global sections of $\mathcal{O}_{\mathbb{P}W}(m)$ is trivial when $m < 0$ and is isomorphic to the vector space of homogeneous polynomial functions of degree m on W when $m \geq 0$.

4. Let V be a complex vector space and let $\Lambda \subseteq V$ be a lattice. Consider the real dual $V^\vee = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$ and its subset

$$\Lambda^\vee := \{\phi \in V^\vee \,|\, \phi(\lambda) \in \mathbb{Z}, \forall \lambda \in \Lambda\}.$$

a) Prove that $f \mapsto \mathrm{Im}(f)$ defines an isomorphism $\mathrm{Hom}_{\bar{\mathbb{C}}}(V, \mathbb{C}) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$, endowing $V^\vee$ with the structure of a $\mathbb{C}$-vector space. Check that $\Lambda^\vee$ is a lattice of $V^\vee$. We call $X^\vee := V^\vee/\Lambda^\vee$ the *dual torus of* $X = V/\Lambda$.

Please turn over!

b) Prove that if X is an abelian variety, so is $X^\vee$.

c) Prove that if X admits a nonconstant meromorphic function, so does $X^\vee$.

d) Let $X = \mathbb{C}^g/(\mathbb{Z}^g + \Omega\mathbb{Z}^g)$. For which $\Omega^\vee$ can we write $X^\vee = \mathbb{C}^g/(\mathbb{Z}^g + \Omega^\vee\mathbb{Z}^g)$? How do parts **b)** and **c)** translate in terms of Ω and $\Omega^\vee$?