# Exercise Sheet 5

**1.** Consider the complex torus $X = \mathbb{C}^g/\Omega\mathbb{Z}^g \oplus \mathbb{Z}^g$. The Néron-Severi group $\mathrm{NS}(X)$ can be considered as the group of alternating forms on $\mathbb{R}^{2g}$, integer-valued on $\mathbb{Z}^{2g}$, which induce a Hermitian form via the $\mathbb{R}$-linear isomorphism $\mathbb{R}^{2g} \xrightarrow{\sim} \mathbb{C}^g$ sending $x \mapsto (\Omega|\mathbb{I}_g)x$. Let $E$ be an alternating form on $\mathbb{R}^{2g}$ with matrix

$$\begin{pmatrix} A & B \\ -{}^tB & C \end{pmatrix} \in M_{2g}(\mathbb{R}).$$

Show that the following conditions are equivalent:

(i) $E \in \mathrm{NS}(X)$

(ii) $A, B, C \in M_g(\mathbb{Z})$ and $A - BZ + {}^tZ{}^tB + {}^tZCZ = 0$.

Use this to find the rank of $\mathrm{NS}(X)$ when $X$ is a very general torus of dimension $g \geq 2$.

**2.** Let $X$ be a complex torus of dimension $g$. The *Picard number* $\varrho(X)$ of $X$ is by definition the rank of $\mathrm{NS}(X)$. Show that $\varrho(X) \leq h^{1,1}(X) = g^2$.

**3.** (*Theorem of the Cube*)

**a)** Let $X_1$, $X_2$ and $X_3$ be complex tori and $L$ a line bundle on $X_1 \times X_2 \times X_3$ such that the restrictions of $L$ to $X_1 \times X_2 \times \{0\}$, $X_1 \times \{0\} \times X_3$ and $\{0\} \times X_2 \times X_3$ are trivial. Show that $L$ is trivial. (*Hint:* use canonical factors.)

**b)** Generalise this to products of $n > 3$ tori.

**4.** Let $X$ be a complex torus and $n \neq 0$ an integer. Denote by $n_X : X \longrightarrow X$ the map $x \mapsto n \cdot x$. Prove that the endomorphism $L \mapsto n_X^* L$ of $\mathrm{Pic}(X)$ induces the $n$-th power map on $X^\vee$ and the $n^2$-th power map on $\mathrm{NS}(X)$.