Exercise Sheet 6

1. Let ω be an alternating non-degenerate integral bilinear form on a free $\mathbb{Z}$-module Λ. Prove that Λ has even rank and that there exists a $\mathbb{Z}$-basis $\mathcal{B}$ of Λ for which the matrix of ω is of the form

$$\begin{pmatrix} 0 & \Delta \\ -\Delta & 0 \end{pmatrix},$$

where $\Delta = \operatorname{diag}(d_1, \dots, d_g)$ and the d_j are positive integers satisfying $d_j | d_{j+1}$. Show that those integers depend only on Λ and ω.

2. (*Morphisms to* $\mathbb{P}^N$)[1] Let X be a complex manifold. Consider the line bundle $\mathcal{O}(1)$ on $\mathbb{P}^N$ with sections $x_0, \dots, x_N$. Show:

a) If $\phi : X \longrightarrow \mathbb{P}^N$ is a morphism of complex manifolds, then $\phi^*(\mathcal{O}(1))$ is a line bundle on X, generated by the global sections $\phi^*(x_0), \dots, \phi^*(x_N)$.

b) Conversely, if $\mathcal{L}$ is a line bundle on X and $s_0, \dots, s_N \in \Gamma(X, \mathcal{L})$ are global sections generating $\mathcal{L}$, then there exists a unique morphism $\phi : X \longrightarrow \mathbb{P}^N$ of complex manifolds such that $\mathcal{L} \cong \phi^*(\mathcal{O}(1))$ and $s_j = \phi^*(x_j)$ for each $j = 0, \dots, N$ under this isomorphism.

3. (*Finding equations*)

a) Question: Let X be an abelian variety and $\mathcal{L}$ a line bundle generated by its global sections, so that we have a morphism $\phi : X \longrightarrow \mathbb{P}^N$. Look at the projective variety $\phi(X)$. How can its polynomial equations be found in relation to theta functions?

b) Let $\Lambda \subseteq \mathbb{C}$ be a lattice, and consider the elliptic curve $E = \mathbb{C}/\Lambda$. Consider the Riemann theta functions with respect to Λ (see Exercise Sheet 3)

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (z) := \sum_{m \in \mathbb{Z}} e^{2\pi i \left(\frac{1}{2}\tau (m+a)^2 + (m+a)(z+b) \right)},$$

and for $j, k \in \{0, 1\}$ define

$$\vartheta_{jk} := \vartheta \begin{bmatrix} j/2 \\ k/2 \end{bmatrix}.$$

[1] For a more general treatment, see R. Hartshorne, *Algebraic Geometry*, II, Theorem 7.1.

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Look at the map $E \longrightarrow \mathbb{P}^2$ sending $z \mapsto (\vartheta_{00}(z)\vartheta_{11}(z)^2, \vartheta_{10}(z)\vartheta_{01}(z)\vartheta_{11}(z), \vartheta_{00}(z)^3)$. One can prove that it induces an isomorphism from E onto the smooth cubic with homogeneous equation

$$Y^2 Z = X(\alpha X - \beta Z)(\beta Z + \alpha Z).$$

Look at O. Debarre, *Complex Tori and Abelian Varieties*, Proposition 2.12 and try to fill up the details of the proof. Explain what is happening.

4. Let $u : X \longrightarrow Y$ be a morphism of complex tori, and let L be a line bundle on Y. Show that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\quad u \quad} & Y \\ \big\downarrow{\scriptstyle \phi_{u^*\mathcal{L}}} & & \big\downarrow{\scriptstyle \phi_{\mathcal{L}}} \\ X^{\vee} & \xleftarrow{\quad u^{\vee} \quad} & Y^{\vee}. \end{array}$$