Exercise Sheet 7

1. Let X_1 and X_2 be abelian varieties and $\mathcal{L}$ an ample line bundle on $X_1 \times X_2$. We wish to establish the inequality

$$\dim H^0(X_1 \times X_2, \mathcal{L}) \leq \dim H^0(X_1, \mathcal{L}|_{X_1}) \dim H^0(X_2, \mathcal{L}|_{X_2}),$$

with equality if and only if $\mathcal{L}$ is isomorphic to $\mathcal{L}|_{X_1} \boxtimes \mathcal{L}|_{X_2}$.

a) For $j = 1, 2$ write $X_j = V_j/\Lambda_j$ and $\mathcal{L}_j := \mathcal{L}|_{X_j}$. Let $\mathcal{B}_j$ be a basis for the lattice Λ_j and $\mathcal{C}_j$ a complex basis for V_j, orthonormal with respect to the Hermitian form associated to $c_1(\mathcal{L}_j)$. Let P_j be the change of basis matrix from the $\mathbb{R}$-basis $\mathcal{B}_j$ to the $\mathbb{R}$-basis $\mathcal{C}_j \sqcup i\mathcal{C}_j$ of V_j. Prove that $\dim H^0(X_j, \mathcal{L}_j) = (\det(P_j))^{-1}$.

b) Let A be an arbitrary square matrix, and let

$$H = \begin{pmatrix} I & A \\ {}^t\!A & I \end{pmatrix}$$

be a positive definite Hermitian matrix. Show that $\det(H) \leq 1$, and that equality holds if and only if $A = 0$.

c) Deduce the desired result.

2. Let $X = V/\Lambda$ be a complex torus and $u : X \longrightarrow X^\vee$ a group homomorphism. Show that the following two conditions are equivalent:

i) there exists a line bundle L on X such that $u = \phi_L$;

ii) $u = u^\vee$.

Do the positive and negative eigenspaces of the involution $u \mapsto u^\vee$ of $\mathrm{Hom}(X, X^\vee)$ have the same rank?

3. We say that a number field K is *totally real* if the image of each embedding of K into $\mathbb{C}$ lies inside $\mathbb{R}$. We say that a number field L is a *CM field* if there is no embedding of L into $\mathbb{R}$ and there exists a totally real field $K \subseteq L$ such that $[L : K] = 2$.

a) Give interesting examples of totally real number fields and CM fields.

Please turn over!

b) Let K be a totally real number field of degree g and $\mathcal{O}_K$ be its ring of integers. Let $\sigma_1, \dots, \sigma_g$ be the embeddings of K into $\mathbb{R}$. Let $\tau_1, \dots, \tau_g$ be complex numbers with positive imaginary part. Show that

$$\Lambda := \{(\sigma_1(p)\tau_1 + \sigma_1(q), \dots, \sigma_g(p)\tau_g + \sigma_g(q)) \mid p, q \in \mathcal{O}_K\} \subseteq \mathbb{C}^g$$

is a lattice and that $X = \mathbb{C}^g/\Lambda$ is an abelian variety.

c) Show that the algebra $\mathrm{End}_{\mathbb{Q}}(X)$ contains K.

d) Let L be a CM-field. We call a *CM-type* a tuple $(\xi_1, \dots, \xi_g)$ of pairwise non-conjugate complex embeddings. Find an analog of **b)** and **c)** in this situation.