Exercise Sheet 9

1. Show that for a "very general" complex torus we have $\mathrm{End}(X) = \mathbb{Z}$. Does the same hold for a "very general" abelian variety?

2. Let $(X, \mathcal{L}_0)$ be a polarized abelian variety. Prove that

$$\gamma : \mathrm{NS}(X)_{\mathbb{Q}} \longrightarrow \mathrm{End}(X)_{\mathbb{Q}}$$
$$\mathcal{L} \longmapsto \phi_{\mathcal{L}_0}^{-1} \circ \phi_{\mathcal{L}}$$

is a $\mathbb{Q}$-linear injective map. Prove that the image of γ consists of the symmetric elements of $\mathrm{End}(X)_{\mathbb{Q}}$.

3. Let X be an abelian variety of dimension g and let f be an automorphism of X of order n. Show that $\phi(n) \leq 2g$, where ϕ is the Euler function.

4. Let k be a number field, and suppose that there exists an automorphism $\sigma \in \mathrm{Aut}(k)$ of order 2 such that the trace form $(x, y) \mapsto \mathrm{tr}(\sigma(x)y)$ on K is positive definite. Prove that k is a CM-field in the following way: the subfield k^σ is totally real and k/k^σ is an imaginary quadratic extension.

5. Problem: Let $\mathbb{H}_{\mathbb{Q}}$ be the $\mathbb{Q}$-algebra of Hamilton quaternions. It can be seen as a $\mathbb{Q}$-subalgebra of $\mathrm{M}_{2,2}(\mathbb{Q})$. Let $\mathbb{H}$ be the integral quaternions and let $V := \mathrm{M}_{2,2}(\mathbb{Q}) \otimes \mathbb{C}$. Can you complete $\mathbb{H}$ into a lattice Λ of V in such a way that $X := V/\Lambda$ is an abelian variety with $\mathrm{End}(X)_{\mathbb{Q}} \supseteq \mathbb{H}_{\mathbb{Q}}$? How often will we obtain $\mathrm{End}(X)_{\mathbb{Q}} = \mathbb{H}_{\mathbb{Q}}$?