Exercise Sheet 1

Exercise 1

Show that $\mathrm{Sp}(2n, \mathbb{R})$ is a regular submanifold of $\mathrm{GL}(2n, \mathbb{R})$ of dimension $n(2n+1)$.

Exercise 2

Verify Lemma 1.4 of the lecture.

Exercise 3

Show that the tangent bundle $\mathrm{T}M$ of a smooth manifold M is a vector bundle in the sense of Definition 1.6 of the lecture.

Exercise 4

(Quaternion Algebra). Let $\mathbb{H} = \mathbb{R}\,\mathbb{1} + \mathbb{R}\,i + \mathbb{R}\,j + \mathbb{R}\,k$ be a four-dimensional real vector space with basis $(\mathbb{1}, i, j, k)$. On $\mathbb{H}$ we define a product by defining it on the basis elements and extending it distributively to $\mathbb{H}$. Set

(i) $\mathbb{1}$ to be the unit,

(ii) $i^2 = j^2 = -1$ and

(iii) $ij = -ji = k$.

Show that this definition turns $\mathbb{H}$ into an associative $\mathbb{R}$-algebra in which every non-zero element has an inverse. Now, for $x = x_0\mathbb{1} + x_1 i + x_2 j + x_3 k$ define $\overline{x} = x_0\mathbb{1} - x_1 i - x_2 j - x_3 k$. Show that $x \cdot \overline{x} = (x_0^2 + x_1^2 + x_2^2 + x_3^2) \cdot \mathbb{1}$. Finally, define $S^3(1) := \{x \in \mathbb{H} \mid x \cdot \overline{x} = 1\}$ and show that $S^3(1)$ is a group with respect to multiplication. Also, show that $S^3(1)$ is parallelizable.