Exercise Sheet 10

Exercise 1

Let (M,g) be a Riemannian manifold with Riemannian curvature tensor R. In local coordinates,

$$R(\partial_i,\partial_j)\partial_k = \sum_{l=1}^{m} R_{ijk}\partial_l.$$

Determine R_{ijk}^l in terms of the Christoffel symbols.

Solution sketch

Recall that $R(X,Y)Z = \nabla_Y\nabla_X Z - \nabla_X\nabla_Y Z + \nabla_{[X,Y]}Z$ and $\nabla_{\partial_i}\partial_j = \sum_k \Gamma_{ij}^k\partial_k$. In particular,

$$\begin{aligned}
\nabla_{\partial_j}\nabla_{\partial_i}\partial_k &= \nabla_{\partial_j}\left(\sum_l \Gamma_{ik}^l\partial_l\right)\\
&= \sum_l \nabla_{\partial_j}(\Gamma_{ik}^l\partial_l)\\
&= \sum_l \left(\partial_j\Gamma_{ik}^l\partial_l + \Gamma_{ik}^l\nabla_{\partial_j}\partial_l\right)\\
&= \sum_l (\partial_j\Gamma_{ik}^l)\partial_l + \sum_r\sum_l \Gamma_{ik}^l\Gamma_{jl}^r\partial_r\\
&= \sum_l (\partial_j\Gamma_{ik}^l)\partial_l + \sum_l\sum_r \Gamma_{ik}^r\Gamma_{jr}^l\partial_l\\
&= \sum_l \left(\partial_j\Gamma_{ik}^l + \sum_r \Gamma_{ik}^r\Gamma_{jr}^l\right)\partial_l
\end{aligned}$$

The expression for $\nabla_{\partial_i}\nabla_{\partial_j}\partial_k$ arises through swapping i and j. Since $[\partial_i,\partial_j]=0$ for all $i,j\in\{1,\dots,m\}$ we conclude

$$\begin{aligned}
R(\partial_i,\partial_j)\partial_k &= \nabla_{\partial_j}\nabla_{\partial_i}\partial_k - \nabla_{\partial_i}\nabla_{\partial_j}\partial_k\\
&= \sum_l \left((\partial_j\Gamma_{ik}^l - \partial_i\Gamma_{jk}^l) + \sum_r\left(\Gamma_{ik}^r\Gamma_{jr}^l - \Gamma_{jk}^r\Gamma_{ir}^l\right)\right)\partial_l.
\end{aligned}$$

Exercise 2

Compute the Riemannian curvature tensor for the hyperbolic plane and determine the (constant) value of its sectional curvature.

Solution sketch

Recall from the lecture that with respect to the global chart $(\mathbb{H}^2,\mathrm{id})$, the Christoffel symbols are given by

$$\Gamma_{11}^1(z) = \Gamma_{12}^2(z) = \Gamma_{22}^1(z) = 0,\ \Gamma_{11}^2(z) = \frac{1}{y} \text{ and } \Gamma_{12}^1(z) = \Gamma_{22}^2(z) = -\frac{1}{y}.$$

They determine the the Riemannian curvature tensor by Exercise 1. To compute the sectional curvature of $\mathbb{H}^2$ we compute

$$K_p(E) = K(\partial_1, \partial_2) = \frac{\langle R(\partial_1, \partial_2)\partial_1, \partial_2\rangle}{|\partial_1 \wedge \partial_2|^2} = -1$$

since

$$\begin{aligned} R(\partial_1, \partial_2)\partial_2 &= \left(\partial_2\Gamma_{11}^2 - \Gamma_{21}^1\Gamma_{11}^2 + \Gamma_{11}^2\Gamma_{22}^2\right)\partial_2 \\ &= \left(-\frac{1}{y^2} + \frac{1}{y^2} - \frac{1}{y^2}\right)\partial_2 \\ &= -\frac{1}{y^2}\partial_2. \end{aligned}$$

Exercise 3

Let N and M be Riemannian manifolds and let $h : N \to M$ be a smooth map. Further, let $h^*(\mathrm{T}M)$ be the pullback of $\mathrm{T}M$ under h. Show the existence and uniqueness of a bilinear map

$$\nabla^h : \Gamma(\mathrm{T}N) \times \Gamma(h^*(\mathrm{T}M)) \to \Gamma(h^*\mathrm{T}M)$$

satisfying Proposition 3.12 of the lecture.

Solution sketch

We proceed as in the case of the covariant derivative along curves. If (U, φ) is a chart of M and $V := h^{-1}(U)$ then for $Y \in \Gamma(h^*(\mathrm{T}M))$ and $y \in V$ we have

$$Y(y) = \sum_{i=1}^m Y_i(y) h^*(\partial_i)(y).$$

Assuming that ∇^h with the asserted properties exists we thus have

$$\nabla_X^h Y(y) = \sum_{i=1}^m X(Y_i)(y)\partial_i(h(y)) + \sum_{i=1}^m Y_i(y)\left(\nabla_X^h \partial_i\right)(h(y))$$

Using

$$(\nabla_X^h \partial_i)(h(y)) = h^*\left(\nabla_{\overline{X}}\partial_i\right)(y) = \left(\nabla_{D_y h(X(y))}\partial_i\right)(h(y))$$

we thus have

$$\nabla_X^h Y(y) = \sum_{i=1}^m X(Y_i)(y)\partial_i(h(y)) + \sum_{i=1}^m Y_i(y)\left(\nabla_{D_y h(X(y))}\partial_i\right)(h(y))$$

which shows uniqueness. As before, the above can be used to define $\nabla^h_X Y$ locally from which we deduce the asserted properties: Let $f \in C^\infty(N)$. Then

$$\begin{aligned}
\nabla^h_{fX} Y(y) &= \sum_{i=1}^m (fX)(Y_i)(y)\partial_i(h(y)) + \sum_{i=1}^m Y_i(y)\left(\nabla_{D_y h((fX)(y))}\partial_i\right)(h(y)) \\
&= \sum_{i=1}^m (fX)(Y_i)(y)\partial_i(h(y)) + \sum_{i=1}^m Y_i(y)\left(\nabla_{f(y)D_y h((X)(y))}\partial_i\right)(h(y)) \\
&= \sum_{i=1}^m (fX)(Y_i)(y)\partial_i(h(y)) + f(y)\sum_{i=1}^m Y_i(y)\left(\nabla_{D_y h((X)(y))}\partial_i\right)(h(y)) \\
&= f\nabla^h_X Y(y)
\end{aligned}$$

as well as $\nabla^h_X(fY)(y)$ equalling

$$\begin{aligned}
&\sum_{i=1}^m X(fY_i)(y)\partial_i(h(y)) + \sum_{i=1}^m fY_i(y)\left(\nabla_{D_y h(X(y))}\partial_i\right)(h(y)) \\
&= \sum_{i=1}^m X(fY_i)(y)\partial_i(h(y)) + \sum_{i=1}^m fY_i(y)\left(\nabla_{D_y h(X(y))}\partial_i\right)(h(y)) \\
&= \sum_{i=1}^m (Y_i X(f)(y) + fX(Y_i(y)))\partial_i(h(y)) + \sum_{i=1}^m fY_i(y)\left(\nabla_{D_y h(X(y))}\partial_i\right)(h(y)) \\
&= X(f)Y(y) + f(y)\sum_{i=1}^m X(Y_i)(y)\partial_i(h(y)) + \sum_{i=1}^m fY_i(y)\left(\nabla_{D_y h(X(y))}\partial_i\right)(h(y)) \\
&= X(f)Y + \nabla^h_X Y.
\end{aligned}$$

Exercise 4

Prove Proposition 3.13 of the lecture. Consider the following hint: Prove that $\nabla^h_X \overline{Y} - \nabla^h_Y \overline{X} - \overline{[X,Y]}$ is $C^\infty(N)$-linear in both variables and conclude by evaluating on coordinate vector fields.

Solution sketch

As an example, we consider the first assertion, namely

$$\nabla^h_X \overline{Y} - \nabla^h_Y \overline{X} = \overline{[X,Y]}$$

for $X, Y \in \Gamma(\mathrm{T}N)$. To see that the expression

$$\nabla^h_X \overline{Y} - \nabla^h_Y \overline{X} - \overline{[X,Y]}$$

is $C^\infty(N)$-linear in the X-variable, let $f \in C^\infty(N)$ and compute

$$\begin{aligned}
&\nabla^h_{fX}\overline{Y} - \nabla^h_Y \overline{fX} - \overline{[fX,Y]} \\
&= f\nabla^h_X \overline{Y} - \nabla^h_Y f\overline{X} - \overline{[fX,Y]} \\
&= f\nabla^h_X \overline{Y} - (Y(f)\overline{X} + f\nabla^h_Y \overline{X}) - (f\overline{[X,Y]} - Y(f)\overline{X}) \\
&= f(\nabla^h_X \overline{Y} - \nabla^h_Y \overline{X} - \overline{[X,Y]}).
\end{aligned}$$

Now, let $X, Y \in \Gamma(\mathrm{T}N)$ be coordinate vector fields. Then $[X, Y] = 0$ and hence $\overline{[X, Y]} = 0$. Therefore,

$$\begin{aligned}
\nabla^h_X \overline{Y} &= \sum_{i=1}^m X(\overline{Y}_i)\partial_i + \sum_{i=1}^m Y_i \nabla_{Dh(X)} \partial_i \\
&= \sum_{i=1} Y(\overline{X}_i)\partial_i + \sum_{i=1}^m \overline{Y}_i \nabla_{\sum_j \overline{X}_j \partial_j} \partial_i \\
&= \sum_{i=1} Y(\overline{X}_i)\partial_i + \sum_{i=1}^m \overline{Y}_i \nabla_{\sum_j \overline{X}_j \partial_j} \partial_i \\
&= \sum_{i=1} Y(\overline{X}_i)\partial_i + \sum_{i=1}^m \overline{Y}_i \sum_j \overline{X}_j \nabla_{\partial_i} \partial_j \\
&= \sum_{i=1} Y(\overline{X}_i)\partial_i + \sum_{j=1}^m \overline{X}_j \nabla_{\sum_i \overline{Y}_i \partial_i} \partial_j \\
&= \nabla^h_Y \overline{X}
\end{aligned}$$

which is the assertion.