Exercise Sheet 11

The goal of this exercise sheet is to prove a Hopf-Rinow type theorem in a purely metric setting, that is, without any smooth structure involved.

Definitions. Let (X, d) be a metric space. A *path* in X is a continuous map c from a compact interval $[a, b] \subseteq \mathbb{R}$ to X. A *geodesic path* joining $x \in X$ to $y \in X$ is a path $c : [0, l] \subseteq \mathbb{R}$ such that $c(0) = x$, $c(l) = y$ and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. The space (X, d) is a *geodesic* if every two points in X can be joined by a geodesic path. The *concatenation* of two paths $c_1 : [a_1, b_1] \to X$ and $c_2 : [a_2, b_2] \to X$ with $c_1(b_1) = c_2(a_2)$ is the path $c : [a_1, b_1+b_2-a_2] \to X$ defined by $c(t) := c_1(t)$ if $t \in [a_1, b_1]$ and $c(t) := c_2(t + a_2 - b_1)$ if $t \in [b_1, b_1 + b_2 - a_2]$. The *length* $l(c)$ of a path $c : [a, b] \to X$ is

$$l(c) = \sup_{(t_0,\dots,t_n)} \sum_{i=0}^{n-1} d(c(t_i), c(t_{i+1}))$$

where $(t_0, \dots, t_n) \in \mathbb{R}^{n+1}$ satisfies $a = t_0 \leq t_1 \leq \dots \leq t_n = b$. The length of c is either a non-negative number or it is infinite. In the former case, c is *rectifiable*. The space (X, d) is a *length space* if the distance between every pair of points in X is equal to the infimum of the length of rectifiable curves joining them.

Statements. Using the above definitions, prove the following.

Lemma. Let (X, d) be a metric space and let $c : [a, b] \to X$ be a path. Prove:

(i) We have $l(c) \geq d(c(a), c(b))$ and $l(c) = 0$ if and only if c is a constant map.

(ii) Let $\varphi : [a', b'] \to [a, b]$ be weakly monotonic. Then $l(c) = l(c \circ \varphi)$.

(iii) If c is the concatenation of two paths c_1 and c_2 then $l(c) = l(c_1) + l(c_2)$.

(iv) Define $\overline{c} : [a, b] \to X$ by $\overline{c}(t) = c(b + a - t)$. Then $l(\overline{c}) = l(c)$.

(v) If c is rectifiable of length l then the function $\lambda : [a, b] \to [0, l]$ defined by $\lambda(t) := l(c|_{[a,t]})$ is continuous and weakly monotonic.

(vi) If c and λ are as in (v), then there is a unique path $\widetilde{c} : [0, l] \to X$ such that $\widetilde{c} \circ \lambda = c$ and $l(\widetilde{c}|_{[0,t]}) = t$ for all $t \in [0, l]$.

(vii) Lower semicontinuity: Let $(c_n)_n$ be a sequence of paths from $[a, b]$ to X converging uniformly to a path c. If c is rectifiable, then for every $\varepsilon > 0$ there exists an integer $N(\varepsilon)$ such that $l(c) \leq l(c_n)+\varepsilon$ whenever $n > N(\varepsilon)$.

Theorem (Hopf-Rinow). Let (X, d) be a length space. If (X, d) is complete and locally compact Hausdorff, then

(i) every closed bounded subset of X is compact, and

(ii) X is a geodesic space.

Hint: For (i), it suffices to prove that closed balls around a fixed point $a \in X$ are compact. Consider the set $\{r \in \mathbb{R} \mid \{x \in X \mid d(a, x) \leq r\}$ is compact$\}$. For part (ii), recall the Arzelà-Ascoli theorem.