Exercise Sheet 12

Let (M, g) be a complete connected Riemannian manifold and let $(\widetilde{M}, \widetilde{g})$ be the universal covering where $\widetilde{g}$ is defined so that $p : \widetilde{M} \to M$ is a Riemannian covering. Further, let Γ be the group of deck transformations of $\widetilde{M}$ which is isomorphic to $\pi_1(M)$. Finally, let d and $\widetilde{d}$ be the Riemannian distances on M and $\widetilde{M}$ respectively.

Given a group Γ which is generated by a finite set S, let

$$\|\gamma\|_S := \min\{n \in \mathbb{N} \mid \gamma \text{ is a product of } n \text{ elements in } S\}.$$

Exercise 1

Show that for $x, y \in \widetilde{M}$ we have $d(p(x), p(y)) = \min_{\gamma \in \Gamma} \widetilde{d}(x, \gamma y)$.

Exercise 2

Fix a basepoint $x_0 \in \widetilde{M}$ and let

$$D := \{x \in \widetilde{M} \mid d(x_0, x) \leq d(\gamma x_0, x) \ \forall \gamma \in \Gamma\}.$$

Show that D is closed and that $\bigcup_{\gamma \in \Gamma} \gamma D = \widetilde{M}$.

Exercise 3

Show that M is compact if and only if D is compact.

Exercise 4

Set $S := \{\gamma \in \Gamma \mid \operatorname{dist}(D, \gamma D) \leq 1\}$. Show that S is finite if D is compact.

Exercise 5

Assume that D is compact. Show that every $\gamma \in \Gamma$ can be written as a product $\gamma = s_1 \cdots s_n$ ($s_i \in S \ \forall i \in \{1, \ldots, n\}$) with $n \leq d(x_0, \gamma x_0) + 1$.

Exercise 6

Let M be a complete, connected and compact Riemannian manifold. Show that $\pi_1(M)$ is finitely generated and that given a basepoint $x_0 \in \widetilde{M}$ there exists a finite generating set S and constants λ, c such that

$$\lambda^{-1}\|\gamma\|_S - c \leq \widetilde{d}(\gamma x_0, x_0) \leq \lambda\|\gamma\|_S + c.$$