Exercise Sheet 12

Let (M, g) be a complete connected Riemannian manifold and let $(\widetilde{M}, \widetilde{g})$ be the universal covering where $\widetilde{g}$ is defined so that $p : \widetilde{M} \to M$ is a Riemannian covering. Further, let Γ be the group of deck transformations of $\widetilde{M}$ which is isomorphic to $\pi_1(M)$. Finally, let d and $\widetilde{d}$ be the Riemannian distances on M and $\widetilde{M}$ respectively.

Given a group Γ which is generated by a finite set S, let

$$\|\gamma\|_S := \min\{n \in \mathbb{N} \mid \gamma \text{ is a product of } n \text{ elements in } S\}.$$

Exercise 1

Show that for $x, y \in \widetilde{M}$ we have $d(p(x), p(y)) = \min_{\gamma \in \Gamma} \widetilde{d}(x, \gamma y)$.

Solution sketch

Since M is complete the distance of $p(x)$ and $p(y)$ is realized by a geodesic c in M. Since p is a local isometry, the length of c is the same as the length of any lift of c startig at x. Since $p^{-1}(y) = \Gamma y$, the endpoint of such a lift is contained in Γy and the assertion follows by the definition of $\widetilde{d}$.

Exercise 2

Fix a basepoint $x_0 \in \widetilde{M}$ and let

$$D := \{x \in \widetilde{M} \mid d(x_0, x) \leq d(\gamma x_0, x)\ \forall \gamma \in \Gamma\}.$$

Show that D is closed and that $\bigcup_{\gamma \in \Gamma} \gamma D = \widetilde{M}$.

Solution sketch

To see that D is closed, let $(y_n)_n$ be a sequence of points in D converging to $y \in \widetilde{M}$. Given $\varepsilon > 0$, we have

$$d(x_0, y) \leq d(x_0, y_n) + d(y_n, y) \leq d(\gamma x_0, x) + \varepsilon$$

for large enough n and all $\gamma \in \Gamma$. Hence $y \in D$ since ε is arbitrary. For the assertion $\bigcup_{\gamma \in \Gamma} \gamma D = \widetilde{M}$, let $x \in M$ and pick $\gamma_0 \in \Gamma$ such that $d(x_0, \gamma_0 x)$ is minimal. We claim that $\gamma_0 x \in D$. If not, there is $\gamma \in \Gamma$ such that

$$d(x_0, \gamma_0 x) > d(\gamma x_0, \gamma_0 x_0).$$

Since Γ acts by isometries on $\widetilde{M}$ this implies $d(x_0, \gamma_0 x) > d(x_0, \gamma^{-1}\gamma_0 x)$ in contradiction to minimality of γ_0.

Exercise 3

Show that M is compact if and only if D is compact.

Solution sketch

Assume that D is compact. By Exercise 2, every point of M has a preimage in D. Hence $p : D \to M$ is surjective and therefore M is compact. Conversely, assume that M is compact. Then M is bounded by continuity of the metric and hence so is D by Exercise 1. Since M is complete, so is $\widetilde{M}$ as seen in the lecture and therefore $D \subseteq \widetilde{M}$ is compact as a closed and bounded set by Hopf-Rinow.

Exercise 4

Set $S := \{\gamma \in \Gamma \mid \operatorname{dist}(D, \gamma D) \leq 1\}$. Show that S is finite if D is compact.

Solution sketch

Consider the set $K := \{x \in \widetilde{M} \mid \operatorname{dist}(x, D) \leq 2\}$. Then K is closed and bounded and hence compact by Hopf-Rinow. Since Γ acts properly discontinuously on $\widetilde{M}$ we conclude that $T := \{\gamma \in \Gamma \mid K \cap \gamma K \neq \emptyset\}$ is finite. The assertion then follows since $S \subseteq T$.

Exercise 5

Assume that D is compact. Show that every $\gamma \in \Gamma$ can be written as a product $\gamma = s_1 \cdots s_n$ ($s_i \in S\ \forall i \in \{1, \ldots, n\}$) with $n \leq d(x_0, \gamma x_0) + 1$.

Solution sketch

Let $\gamma \in \Gamma$ and let c be a geodesic in $\widetilde{M}$ connecting x_0 to γx_0. We define $n := \lfloor d(x_0, \gamma x_0) \rfloor + 1$. Set $x_n := \gamma x_0$ and for $1 \leq i \leq n-1$, pick $x_i \in \widetilde{M}$ on c such that $d(x_{i-1}, x_i) = 1$. Then $d(x_{n-1}, x_n) \leq 1$. Since $\bigcup_{\gamma \in \Gamma} \gamma D = \widetilde{M}$ there are $\gamma_i \in \Gamma$ with $x_i \in \gamma_i D$ for $0 \leq i \leq n$. Choose $\gamma_0 = e$ and $\gamma_n = \gamma$. Set $s_i := \gamma_{i-1}^{-1}\gamma_i$ for all $1 \leq i \leq n$. Then $\gamma = \prod_{i=1}^n s_i$ and $s_i \in S$ since

$$\operatorname{dist}(D, s_i D) = \operatorname{dist}(D, \gamma_{i-1}^{-1}\gamma_i D) = \operatorname{dist}(\gamma_{i-1} D, \gamma_i D) \leq d(x_{i-1}, x_i) = 1.$$

Exercise 6

Let M be a complete, connected and compact Riemannian manifold. Show that $\pi_1(M)$ is finitely generated and that given a basepoint $x_0 \in \widetilde{M}$ there exists a finite generating set S and constants λ, c such that

$$\lambda^{-1}\|\gamma\|_S - c \leq \widetilde{d}(\gamma x_0, x_0) \leq \lambda \|\gamma\|_S + c.$$

Solution sketch

Let x_0, D and S be as above. By the definition of $\|-\|_S$ and Exercise 5 we have

$$\|\gamma\|_S - 1 \leq \widetilde{d}(x_0, \gamma x_0)$$

for all $\gamma \in \Gamma$. On the other hand, the definitions of S and D imply that $\widetilde{d}(x_0, s x_0) \leq 1 + 2\operatorname{diam}(D)$. Therefore, with $s_0 := \operatorname{id} \in \Gamma$,

$$\widetilde{d}(x_0, \gamma x_0) \leq \sum_{i=1}^{n} \widetilde{d}(s_{i-1} x_0, s_i x_0) \leq (1 + 2\operatorname{diam}(D))\|\gamma\|_S.$$