Exercise Sheet 1

Exercise 1

Show that $\mathrm{Sp}(2n,\mathbb{R})$ is a regular submanifold of $\mathrm{GL}(2n,\mathbb{R})$ of dimension $n(2n+1)$.

Solution sketch

We apply the regular value theorem to the map

$$f : \mathrm{GL}(2n,\mathbb{R}) \to \mathrm{GL}(2n,\mathbb{R}),\ g \mapsto g^T Jg.$$

Given $g \in \mathrm{GL}(2n,\mathbb{R})$ and $Y \in \mathrm{T}_g \mathrm{GL}(2n,\mathbb{R}) \cong M_{2n,2n}(\mathbb{R})$ we have

$$(D_g f)(Y) = Y^T Jg + g^T JY$$

Note that $D_g f$ is anti-symmetric and that every anti-symmetric matrix can be written as $(D_g f)(y)$ for some $Y \in M_{2n,2n}(\mathbb{R})$. Hence f has constant rank and we conclude that $\mathrm{Sp}(2n,\mathbb{R})$ is a regular submanifold of $\mathrm{GL}(2n,\mathbb{R})$ of dimension $(2n)^2 - 2n(2n-1)/2 = n(2n+1)$.

Exercise 2

Verify Lemma 1.4 of the lecture.

Solution sketch

Recall that $C_p = \{(I,c) \mid 0 \in I \subseteq \mathbb{R} \text{ open},\ c : I \to M,\ c(0) = p\}$ and that (I,c) and (J,γ) in C_p are equivalent if and only if $(\varphi \circ c)'(0) = (\varphi \circ \gamma)'(0)$ for some chart (U,φ) of M at p. Now consider the map

$$f : C_p \to \mathcal{A}_p / \sim_p,\ (I,c) \mapsto [(U,\varphi,(\varphi \circ c)'(0))]$$

If $(I,c) \sim_p (J,\gamma)$ then $(\varphi \circ c)'(0) = (\varphi \circ \gamma)'(0)$ and hence $f(I,c) = f(J,\gamma)$. Hence f descends to a map $\widetilde{f} : C_p / \sim_p \to \mathcal{A}_p / \sim_p$. To check injectivity of $\widetilde{f}$, suppose $[(U,\varphi,(\varphi \circ c)'(0))] = [(U,\varphi,(\varphi \circ \gamma)'(0))]$. Then $D_{\varphi(p)}(\varphi\varphi^{-1})(\varphi \circ c)'(0) = (\varphi \circ \gamma)'(0)$, i.e. $(\varphi \circ c)'(0) = (\varphi \circ \gamma)'(0)$ which amounts to equivalence of (I,c) and (J,φ). As to surjectivity, let $\xi \in \mathbb{R}^m$ and consider the map $c : [-a,a] \to M$ given by $t \mapsto \varphi^{-1}(t\xi)$ for suitable $a \in \mathbb{R}$. Then $(\varphi \circ c)'(0) = \xi$ and hence $f(I,c) = [(U,\varphi,\xi)]$.

Exercise 3

Show that the tangent bundle $\mathrm{T}M$ of a smooth manifold M is a vector bundle in the sense of Definition 1.6 of the lecture.

Solution sketch

We exhibit the triple $(\pi, \mathrm{T}M, M)$, where $\pi : \mathrm{T}M \to M$ is given by $(x,v) \mapsto x$, as a vector bundle: The projection π is continuous and smooth by the definition of the topology and atlas on $\mathrm{T}M$. It is furthermore obviously surjective. Let $\{(U_\alpha,\varphi_\alpha) \mid \alpha \in \mathcal{A}\}$ be the atlas of M. Then the chart domains $(U_\alpha)_\alpha$ cover M

and as local trivializations $h_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$ we may set $h_\alpha(x,v) := (x, D_x\varphi(v))$. Given intersecting charts $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β), the map $g_{\alpha\beta} \in \mathrm{GL}(n,\mathbb{R})$ in $h_\alpha h_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^n \to (U_\alpha \cap U_\beta) \times \mathbb{R}^n$ given by $(x,v) \mapsto (x, g_{\alpha\beta}(x)v)$ is given by $D(\varphi\psi^{-1})$.

Exercise 4

(Quaternion Algebra). Let $\mathbb{H} = \mathbb{R}\,\mathbb{1} + \mathbb{R}\,i + \mathbb{R}\,j + \mathbb{R}\,k$ be a four-dimensional real vector space with basis $(\mathbb{1}, i, j, k)$. On $\mathbb{H}$ we define a product by defining it on the basis elements and extending it distributively to $\mathbb{H}$. Set

(i) $\mathbb{1}$ to be the unit,

(ii) $i^2 = j^2 = -\mathbb{1}$ and

(iii) $ij = -ji = k$.

Show that this definition turns $\mathbb{H}$ into an associative $\mathbb{R}$-algebra in which every non-zero element has an inverse. Now, for $x = x_0\mathbb{1} + x_1 i + x_2 j + x_3 k$ define $\overline{x} = x_0\mathbb{1} - x_1 i - x_2 j - x_3 k$. Show that $x \cdot \overline{x} = (x_0^2 + x_1^2 + x_2^2 + x_3^2) \cdot \mathbb{1}$. Finally, define $S^3(1) := \{x \in \mathbb{H} \mid x \cdot \overline{x} = \mathbb{1}\}$ and show that $S^3(1)$ is a group with respect to multiplication. Also, show that $S^3(1)$ is parallelizable.

Solution sketch

The set $S^3(1)$ is the unit sphere in $\mathbb{R}^4$ and is equipped with the usual smooth structure. It constitutes a group with identity element $\mathbb{1}$ and the given multiplication as inverses of elements in $S^3(1)$ are again elements of $S^3(1)$. For this group, multiplication and inversion are smooth maps as multiplication with i, j and k are linear and hence smooth maps of $\mathbb{R}^4$. A trivialization of the tangent bundle of $S^3(1)$ is this given by

$$h : \mathrm{T}S^3 \to S^3(1) \times \mathrm{T}_{\mathbb{1}}S^3(1),\ (x,v) \mapsto (x, D_x L_{x^{-1}} v)$$

where $L_x : S^3(1) \to S^3(1)$ denotes left multiplication by $x \in S^3(1)$.