Exercise Sheet 2

Exercise 1

Let M be a smooth manifold and let $X : M \to \mathrm{T}M$ be a vector field on M. Show that X is smooth if and only if its local expression in every chart is given by smooth functions.

Exercise 2

Let M be a smooth manifold. Given a smooth vector field $X \in \Gamma(\mathrm{T}M)$, let $L_X \in \mathrm{Der}(C^\infty(M))$ be the associated derivation given by $L_X(f)(p) := X(p)(f)$. Show that the map $\Gamma(\mathrm{T}M) \to \mathrm{Der}(C^\infty(M))$ is an isomorphism.

Exercise 3

Prove Corollary 1.22 of the lecture.

Exercise 4

Let M be a smooth manifold and let X, Y and Z be smooth vector fields on M. Further, let θ_t denote the local group of diffeomorphisms associated to Z. Which identity do you get when taking the derivative at $t = 0$ of

$$(\theta_t)_*([X,Y]) = [(\theta_t)_*X, (\theta_t)_*Y]?$$

Exercise 5

Consider $\mathbb{R}^n$ endowed with the scalar product $\langle x, y\rangle := \sum_{i=1}^n x_i y_i$ $(x, y \in \mathbb{R}^n)$ and let $\| - \|$ denote the corresponding norm. For every $A \in M_{n,n}(\mathbb{R})$, set

$$\|A\| := \max_{\substack{v\in\mathbb{R}^n \\ \|v\|\leq 1}} \|Av\|$$

Show that this yields a norm on $M_{n,n}(\mathbb{R})$ satisfying $\|A \cdot B\| \leq \|A\| \cdot \|B\|$ for all $A, B \in M_{n,n}(\mathbb{R})$. Further, given $A \in M_{n,n}(\mathbb{R})$, set

$$\mathrm{Exp}\, A := \sum_{n=0}^{\infty} \frac{A^n}{n!}.$$

Show that Exp coverges uniformly on compact subsets of $M_{n,n}(\mathbb{R})$ as well as all its derivates. Conclude that the map

$$M_{n,n}(\mathbb{R}) \to \mathrm{GL}(n, \mathbb{R}),\ A \mapsto \mathrm{Exp}\, A$$

is smooth. Compute the derivative of this map at $0 \in M_{n,n}(\mathbb{R})$ and show that $[A, B] = 0$ implies $\mathrm{Exp}(A + B) = \mathrm{Exp}(A) \cdot \mathrm{Exp}(B)$.