ETH Zürich | D-MATH | Differential Geometry II
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Exercise Sheet 2

Exercise 1

Let M be a smooth manifold and let $X : M \to \mathrm{T}M$ be a vector field on M. Show that X is smooth if and only if its local expression in every chart is given by smooth functions.

Solution sketch

Given a chart (U, φ) of M, consider the following commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{X} & \mathrm{T}M \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle D\varphi} \\ \varphi(U) & \longrightarrow & \varphi(U) \times \mathbb{R}^m \,. \end{array}$$

The composition $D\varphi \circ X \circ \varphi^{-1}$ is given by $y \mapsto (y, (D_{\varphi^{-1}(y)}\varphi)(X(\varphi^{-1}(y))))$. This map is smooth if and only if each of the m coordinates of $(D_{\varphi^{-1}(y)}\varphi)(X(\varphi^{-1}(y)))$ is smooth in y. Since the local expression of X is given $X(x) = \sum_{i=1}^m X_i(x)\partial_i(x)$ where $\partial_i(x)$ is the derivation of $C^\infty(M)$ at x corresponding to the tangent vector $(D_{\varphi(x)}\varphi^{-1})(e_i)$, this amounts exactly to smoothness of the maps $X_i : U \to \mathbb{R}$ and conversely.

Exercise 2

Let M be a smooth manifold. Given a smooth vector field $X \in \Gamma(\mathrm{T}M)$, let $L_X \in \mathrm{Der}(C^\infty(M))$ be the associated derivation given by $L_X(f)(p) := X(p)(f)$. Show that the map $\Gamma(\mathrm{T}M) \to \mathrm{Der}(C^\infty(M))$ is an isomorphism.

Solution sketch

The fact that L_X is a derivation follows from the fact that $f \mapsto (L_X f)(p)$ is a derivation at $p \in M$.

Injectivity of the map $\Gamma(\mathrm{T}M) \to \mathrm{Der}(C^\infty(M)),\ X \mapsto L_X$ follows from the fact that T_pM and $\mathrm{Der}_p(C^\infty(M))$ are isomorphic.

We now turn to surjectivity: Let $\delta \in \mathrm{Der}(C^\infty(M))$. Then for every $p \in M$, the map $f \mapsto \delta(f)(p)$ is a derivation at p. Hence there is a well-defined vector $X(p) \in \mathrm{T}_pM$ such that $\delta(f)(p) = X(p)(f)$ for all $f \in C^\infty(M)$. Together, these vectors form a vector field X on M. We need to verify that X is smooth. To this end, we argue in local coordinates: Let (U, φ) be a chart of M and let $X(p) = \sum_{i=1}^m X_i(p)\, \partial_i(p)$ $(p \in U)$ be the associated representation of X. Smoothness of X amounts to smoothness of the coefficient functions X_i $(i \in \{1, \ldots, m\})$. For every $f \in C^\infty(M)$, the map $p \mapsto \delta(f)(p) = X(p)f$ is smooth. In particular, the map $p \mapsto \sum_{i=1}^m X_i(p)\partial_i(p)(f)$ is smooth on U. Now recall that

$$\partial_i(p)(f) = \frac{\partial(f \circ \varphi^{-1})}{\partial x_i}(\varphi(p)).$$

Hence, if $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}$ was given by $x \mapsto x_l$ we would obtain $\partial_i(p)(f_l) = \delta_{il}$ and conclude that X_l is smooth. However, there may not be such a function defined on the whole of M. Yet, since it suffices to verify smoothness of the functions X_i ($i \in \{1, \dots, m\}$) locally, we may choose

$$f(x) := \begin{cases} \psi_p(x) \cdot (\mathrm{pr}_l \circ \varphi) & x \in \operatorname{supp} \psi_p \\ 0 & x \notin \operatorname{supp} \psi_p \end{cases}$$

where $\psi_p : M \to \mathbb{R}$ is a smooth function which is identically equal to one on a neighbourhood of $p \in U$ and whose support is contained in U.

Exercise 3

Prove Corollary 1.21 of the lecture.

Solution sketch

Let $x_0 \in M$. In order to deduce local existence and uniqueness of integral curves for X, let (U, φ) be a chart of M at x_0. Then consider the push-forward vector field $\varphi_* X$ on $\varphi(U)$. Using a bump function which is identically equal to one on a neighbourhood of $\varphi(x_0)$ and whose support is contained in $\varphi(U)$ we may extend $\varphi_* X$ to the whole of $\mathbb{R}^m$. In this situation, we are guarenteed local existence, uniqueness and smooth dependence on initial conditions. Post-composing integral curves of $\varphi_* X$ with φ^{-1} produces integral curves of X in a small enough neighbourhood of x_0. We thereby establish the analogous result in the context of manifolds.

Exercise 4

Let M be a smooth manifold and let X, Y and Z be smooth vector fields on M. Further, let θ_t denote the local group of diffeomorphisms associated to Z. Which identity do you get when taking the derivative at $t = 0$ of

$$(\theta_t)_*([X, Y]) = [(\theta_t)_* X, (\theta_t)_* Y]?$$

Solution sketch

The derivative of the above identity at $t = 0$ yields the Jacobi identity. We deduce this from Proposition 1.31 of the lecture which is going to be proven at a later point. Said proposition applies directly to the left hand side:

$$\left.\frac{d}{dt}\right|_{t=0} (\theta_t)_*([X, Y]) = [[X, Y], Z].$$

As to the right hand side, one shows that

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} [(\theta_t)_* X, (\theta_t)_* Y] &= \left[\left.\frac{d}{dt}\right|_{t=0} (\theta_t)_* X, Y \right] + \left[X, \left.\frac{d}{dt}\right|_{t=0} (\theta_t)_* Y \right] \\ &= [[X, Y], Z] + [X, [Y, Z]] \end{aligned}$$

by exhibiting all three terms in local coordinates as developed in the lecture. Hence the assertion.

Exercise 5

Consider $\mathbb{R}^n$ endowed with the scalar product $\langle x, y\rangle := \sum_{i=1}^n x_i y_i$ $(x, y \in \mathbb{R}^n)$ and let $\|-\|$ denote the corresponding norm. For every $A \in M_{n,n}(\mathbb{R})$, set

$$\|A\| := \max_{\substack{v\in\mathbb{R}^n \\ \|v\|\leq 1}} \|Av\|$$

Show that this yields a norm on $M_{n,n}(\mathbb{R})$ satisfying $\|A \cdot B\| \leq \|A\| \cdot \|B\|$ for all $A, B \in M_{n,n}(\mathbb{R})$. Further, given $A \in M_{n,n}(\mathbb{R})$, set

$$\operatorname{Exp} A := \sum_{n=0}^{\infty} \frac{A^n}{n!}.$$

Show that Exp converges uniformly on compact subsets of $M_{n,n}(\mathbb{R})$ as well as all its derivates. Conclude that the map

$$M_{n,n}(\mathbb{R}) \to \mathrm{GL}(n, \mathbb{R}),\ A \mapsto \operatorname{Exp} A$$

is smooth. Compute the derivative of this map at $0 \in M_{n,n}(\mathbb{R})$ and show that $[A, B] = 0$ implies $\operatorname{Exp}(A + B) = \operatorname{Exp}(A) \cdot \operatorname{Exp}(B)$.

Solution sketch

It follows readily from the according properties of the norm on $\mathbb{R}^n$ that we obtain a norm on $M_{n,n}(\mathbb{R})$. Submultiplicativity holds true if $\|B\| = 0$. If $\|B\| \neq 0$ we compute

$$\|AB\| := \max_{\substack{v\in\mathbb{R}^n \\ \|v\|\leq 1}} \|ABv\| = \max_{\substack{v\in\mathbb{R}^n \\ \|v\|\leq 1}} \left\| A\frac{Bv}{\|B\|}\right\| \cdot \|B\| \leq \|A\| \cdot \|B\|.$$

To prove uniform convergence on balls of finite radius, note that

$$\left\| \operatorname{Exp} A - \sum_{n=0}^{N} \frac{A^n}{n!} \right\| = \left\| \sum_{n=N+1}^{\infty} \frac{A^n}{n!} \right\| \leq \sum_{n=N+1}^{\infty} \frac{\|A\|^n}{n!} \leq \sum_{n=N+1}^{\infty} \frac{R^n}{n!}$$

for all $A \in M_{n,n}(\mathbb{R})$ with $\|A\| \leq R$. In order to show the continuity of partial derivatives of Exp, we need to understand the partial derivatives of the map $M_{n,n}(\mathbb{R}) \to M_{n,n}(\mathbb{R}),\ X \mapsto X^n$. For instance,

$$\frac{\partial}{\partial x_{ij}} X^n = \sum_{k_1+k_2=n-1} X^{k_1} E_{ij} X^{k_2}$$

by applying the product rule. Therefore

$$\left\| \frac{\partial}{\partial x_{ij}} X^n \right\| \leq n\|X\|^{n-1}$$

which establishes continuity of the first partial derivatives of Exp. Iterate this to deal with higher order derivatives.

As to the derivative of Exp at $0 \in M_{n,n}(\mathbb{R})$ we have for all $H \in M_{n,n}(\mathbb{R})$:

$$D_0 \operatorname{Exp}(H) = \left.\frac{d}{dt}\right|_{t=0} \operatorname{Exp}(tH) = H$$

Hence $D_0 \operatorname{Exp} = \operatorname{Id}$.

Finally, note that if $[A, B] = 0$ then

$$(A+B)^n = \sum_{k=0}^{n} \binom{n}{k} A^k B^{n-k}$$

by the binomial theorem and hence

$$\begin{aligned}
\operatorname{Exp}(A+B) &= \sum_{n=0}^{\infty} \frac{(A+B)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k_1+k_2=n} \binom{n}{k_1} A^{k_1} B^{k_2} = \\
&= \sum_{n=0}^{\infty} \sum_{k_1+k_2=n} \frac{1}{k_1! k_2!} A^{k_1} B^{k_2} = \sum_{k_1,k_2=0}^{\infty} \frac{A^{k_1}}{k_1!} \frac{B^{k_2}}{k_2!} = \\
&= \left(\sum_{k_1=0}^{\infty} \frac{A^{k_1}}{k_1!} \right) \left(\sum_{k_2=0}^{\infty} \frac{B^{k_2}}{k_2!} \right) \\
&= \operatorname{Exp}(A) \operatorname{Exp}(B)
\end{aligned}$$