Exercise Sheet 3

Exercise 1

For $g \in \mathrm{GL}(n,\mathbb{C})$, define $g^* := \overline{g}^T$. Show that both

$$\mathrm{U}(n) := \{g \in \mathrm{GL}(n,\mathbb{C}) \mid g^* g = \mathrm{Id}\}$$

and

$$\mathrm{SU}(n) := \{g \in \mathrm{U}(n) \mid \det g = 1\}$$

are Lie groups.

Exercise 2

Compute the Lie algebras of $\mathrm{U}(n)$ and $\mathrm{SU}(n)$.

Exercise 3

Let G be a Lie group and let $H \leq G$ be a regular submanifold. Show that $\exp_H : \mathrm{T}_e H \to H$ is given by the restriction of $\exp_G : \mathrm{T}_e G \to G$ to $\mathrm{T}_e H \leq \mathrm{T}_e G$.

Exercise 4

Retain the notation of Exercise 3 and deduce that

$$\mathrm{T}_e H = \{v \in \mathrm{T}_e G \mid \exp_G(tv) \in H \ \forall t \in \mathbb{R}\}.$$

Exercise 5

Let G be a Lie group. Show that the map $\mathrm{Int}(g) : G \to G,\ h \mapsto ghg^{-1}$ is a diffeomorphism of G for all $g \in G$. Furthermore, show that $D_e\mathrm{Int}(g) = \mathrm{Ad}(g)$.