Exercise Sheet 3

Exercise 1

For $g \in \mathrm{GL}(n, \mathbb{C})$, define $g^* := \overline{g}^T$. Show that both

$$\mathrm{U}(n) := \{g \in \mathrm{GL}(n, \mathbb{C}) \mid g^* g = \mathrm{Id}\}$$

and

$$\mathrm{SU}(n) := \{g \in \mathrm{U}(n) \mid \det g = 1\}$$

are Lie groups.

Solution sketch

We consider $\mathrm{GL}(n, \mathbb{C})$ as an open subset of $M_{n,n}(\mathbb{C})$ which comes equipped with the structure of a (real) smooth manifold as a $2n^2$-dimensional real vector space, hence so does $\mathrm{GL}(n, \mathbb{C})$. Applying the constant rank theorem to the map

$$f : \mathrm{GL}(n, \mathbb{C}) \to M_{n,n}(\mathbb{C}),\ A \mapsto A^* A \quad \text{with} \quad D_{\mathrm{Id}} f(X) = X + X^*$$

shows that $\mathrm{U}(n)$ is a regular submanifold of $\mathrm{GL}(n, \mathbb{C})$. Turning to $\mathrm{SU}(n, \mathbb{C})$, consider the restriction of the determinant map to $\mathrm{U}(n)$ and recall that $D_{\mathrm{Id}} \det(X) = \operatorname{tr} X$. Matrix multiplication and inversion are smooth maps as before. Thus $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ are Lie groups.

Exercise 2

Compute the Lie algebras of $\mathrm{U}(n)$ and $\mathrm{SU}(n)$.

Solution sketch

In the case of $\mathrm{U}(n)$ we proceed as in the lecture, that is using Exercise 4:

$$\begin{aligned}
\mathrm{Lie}(\mathrm{U}(n)) &= \{x \in M_{n,n}(\mathbb{C}) \mid \mathrm{Exp}(tx) \in \mathrm{U}(n)\ \forall t \in \mathbb{R}\} \\
&= \{x \in M_{n,n}(\mathbb{C}) \mid \mathrm{Exp}(tx)^* \,\mathrm{Exp}(tx) = \mathrm{Id}\ \forall t \in \mathbb{R}\} \\
&= \{x \in M_{n,n}(\mathbb{C}) \mid x + x^* = 0\}
\end{aligned}$$

As a consequence, we obtain for $\mathrm{SU}(n)$:

$$\begin{aligned}
\mathrm{Lie}(\mathrm{SU}(n)) &= \{x \in \mathrm{Lie}(\mathrm{U}(n)) \mid \det \mathrm{Exp}(tx) = 1\ \forall t \in \mathbb{R}\} \\
&= \{x \in M_{n,n}(\mathbb{C}) \mid x + x^* = 0 \text{ and } \operatorname{tr} x = 0\}.
\end{aligned}$$

Exercise 3

Let G be a Lie group and let $H \leq G$ be a regular submanifold. Show that $\exp_H : \mathrm{T}_e H \to H$ is given by the restriction of $\exp_G : \mathrm{T}_e G \to G$ to $\mathrm{T}_e H \leq \mathrm{T}_e G$.

Solution sketch

Let $x \in \mathrm{T}_e H$ and let $X^H \in \Gamma^{\mathrm{inv}}(\mathrm{T}H)$ denote the associated left-invariant vector field on H. Now consider the natural left-invariant extension X^G of X^H to G. Then the integral curve of X^G through $e \in H \leq G$ with initial condition x stays in H and satisfies the same initial condition as the map $t \mapsto \exp_H(tx)$. Hence $\exp_H = \exp_G |_{\mathrm{T}_e H}$.

Exercise 4

Retain the notation of Exercise 3 and deduce that

$$\mathrm{T}_e H = \{v \in \mathrm{T}_e G \mid \exp_G(tv) \in H \ \forall t \in \mathbb{R}\}.$$

Solution sketch

Exercise 3 readily implies that $\mathrm{T}_e H \subseteq \{v \in \mathrm{T}_e G \mid \exp_G(tv) \in H \ \forall t \in \mathbb{R}\}$. Conversely, suppose that $v\ in \mathrm{T}_e G$ satisfies $\exp_G(tv) \in H$ for all times $t \in \mathbb{R}$. Then $(\mathbb{R}, t \mapsto \exp_G(tv))$ is a pair consisting of an open interval of $\mathbb{R}$ containing 0 and a smooth curve from said interval into H with initial value $e \in H \leq G$ as in the definition of tangent vectors of H at $e \in H$. Hence $v \in \mathrm{T}_e H$.

Exercise 5

Let G be a Lie group. Show that the map $\mathrm{Int}(g) : G \to G,\ h \mapsto ghg^{-1}$ is a diffeomorphism of G for all $g \in G$. Furthermore, show that $D_e\mathrm{Int}(g) = \mathrm{Ad}(g)$.

Solution sketch

The map $\mathrm{Int}(g)$ is a diffeomorphism of G as composition (either way) of the diffeomorphisms L_g and $R_{g^{-1}}$. Now, by definition, we have for $g \in G$ and $x \in \mathrm{T}_e G$:

$$\mathrm{Ad}(g)x := ((R_{g^{-1}})_* X)_e,$$

where $X \in \Gamma^{\mathrm{inv}}(\mathrm{T}G)$ is the left-invariant vector field associated to $x \in \mathrm{T}_e G$, and therefore conclude

$$\mathrm{Ad}(g)x = D_g R_{g^{-1}} X(g) = D_g R_{g^{-1}} D_e L_g x = D_e \mathrm{Int}(g)x.$$