Exercise Sheet 4

Exercise 1

Show that the action of $\mathrm{SL}(2,\mathbb{Z})$ on $H^+ := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ given by

$$\mathrm{SL}(2,\mathbb{Z}) \times H^+ \to H^+, \quad \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z\right) \mapsto \frac{az+b}{cz+d}$$

is properly discontinuous.

Exercise 2

Let Γ be a group acting freely and properly discontinuously on a manifold M by diffeomorphisms. Show that every $x \in M$ admits an open neighbourhood V_x such that $\gamma V_x \cap V_x = \emptyset$ for all $\gamma \in \Gamma$.

Exercise 3

This exercise aims to assist in proving the following statement: Let M be a manifold and let $R \subseteq M \times M$ be an equivalence relation on M. Then there is a unique smooth manifold structure on M/R such that the natural map $p : M \to M/R$ is a submersion if and only if R is a closed submanifold of $M \times M$ and $\mathrm{pr}_1|_R : R \to M$ is a submersion.

(i) Show the necessity part of the above statement.

Suffiency can be proven by following the subsequent list of statements.

(ii) Show that $p : M \to M/R$ is open.

As a consequence of (i), the quotient topology on M/R is Hausdorff. The following two statements now turn the problem into a local one. If U is a subset of M we let $R_U := R \cap (U \times U)$ denote the restriction of R to U. Also, recall that U is *saturated* with respect to R if $U = p^{-1}(p(U))$.

(iii) Assume that $M = \bigcup_{i\in I} U_i$ where $U_i \subseteq M$ is open, saturated with respect to R and such that U_i/R_{U_i} is a manifold with submersive projection. Then X/R is a manifold with submersive projection.

(iv) Assume that $U \subseteq M$ is open, satisfies $p^{-1}(p(U)) = M$ and U/R_U is a manifold with submersive projection. Then X/R is a manifold with submersive projection.

(v) Assume that $M = \bigcup_{i\in I} U_i$ where $U_i \subseteq M$ is an open and such that U_i/R_{U_i} is a manifold with submersive projection. Then X/R is a manifold with submersive projection.

We now show that the local problem phrased in part (iii) has a solution.

(vi) Let $m \in M$. Then there is an open neighbourhood $U \subseteq M$ of $m \in M$, a closed submanifold S of U and a submersion $q : U \to S$ such that for every $x \in U$ the set $[x] \cap U$ intersects S in $q(x)$.

(vii) Retain the notation of part (iv). Then U/R_U is a manifold with submersive projection.