Exercise Sheet 4

Exercise 1

Show that the action of $\mathrm{SL}(2,\mathbb{Z})$ on $H^+ := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ given by

$$\mathrm{SL}(2,\mathbb{Z}) \times H^+ \to H^+, \quad \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z\right) \mapsto \frac{az+b}{cz+d}$$

is properly discontinuous.

Solution sketch

Given $g \in \mathrm{SL}(2,\mathbb{Z})$ and $z \in H^+$, compute

$$\mathrm{Im}(gz) = \frac{\mathrm{Im}(z)}{|cz+d|^2}.$$

It therefore suffices to show that for a small $\varepsilon > 0$, the set

$$\{(c,d) \in \mathbb{Z}^2 \mid |cz+d|^2 \in [1-\varepsilon, 1+\varepsilon]\}$$

is finite. Indeed, let $z = x + iy$ with $y > 0$. Then

$$|cz+d|^2 = (cx+d)^2 + c^2y^2$$

In particular, $c^2y^2 \leq 1+\varepsilon$ which admits only finitely many integer solutions. For any such solution c the inequality $(cx+d)^2 + c^2y^2 \leq 1+\varepsilon$ now only has finitely many integer solutions.

Exercise 2

Let Γ be a group acting freely and properly discontinuously on a manifold M by diffeomorphisms. Show that every $x \in M$ admits an open neighbourhood V_x such that $\gamma V_x \cap V_x = \emptyset$ for all $\gamma \in \Gamma$.

Solution sketch

Since Γ acts properly discontinuously on M there is an open neighbourhood U of $x \in M$ such that

$$\{\gamma \in \Gamma \mid \gamma U \cap U \neq \emptyset\} =: \{\gamma_1, \dots, \gamma_n\}$$

is finite. Using Hausdorfness of M and the fact that Γ acts freely on M, choose open neighbourhoods V_i and W_i of x and $\gamma_i x$ respectively for each $i \in \{1, \dots, n\}$ such that $V_i \cap W_i = \emptyset$. Then $V_i \cap \gamma_i^{-1} W_i$ is an open neighbourhood of $x \in M$ such that $(V_i \cap \gamma_i^{-1} W_i) \cap \gamma_i(V_i \cap \gamma_i^{-1} W_i) = \emptyset$ and we may hence set

$$V_x := U \cap \bigcap_{i=1}^{n} V_i \cap \gamma_i^{-1} W_i.$$

Exercise 3

This exercise aims to assist in proving the following statement: Let M be a manifold and let $R \subseteq M \times M$ be an equivalence relation on M. If R is a closed submanifold of $M \times M$ and $\mathrm{pr}_1|_R : R \to M$ is a submersion then there is a unique smooth manifold structure on M/R such that the natural map $p : M \to M/R$ is a submersion.

Solution sketch

See van den Ban's "Notes on quotients and group actions" and Serre's "Lie algebras and Lie groups", Part II, Chapter III.12.