Exercise Sheet 5

Exercise 1

Equip $\mathbb{R}^n$ with the Riemannian metric associated to the standard scalar product, i.e. for all $x \in \mathbb{R}^n$ and $v, w \in \mathrm{T}_x \mathbb{R}^n$ we set

$$g_x(v,w) = \langle v, w\rangle = \sum_{i=1}^n v_i w_i.$$

Show that the resulting Riemannian distance coincides with the Euclidean.

Exercise 2

Let $\Gamma \leq \mathrm{Iso}(\mathbb{R}^n, \mathrm{can})$ be a Bieberbach group consisting of pure translations. Show that there is a basis $(a_1, \dots, a_n)$ of $\mathbb{R}^n$ such that

$$\Gamma = \{T_a \mid a \in \mathbb{Z}\, a_1 + \cdots + \mathbb{Z}\, a_n\}.$$

Exercise 3

Let Γ be the subgroup of $\mathrm{Iso}(\mathbb{R}^3, \mathrm{can})$ generated by $\gamma_1, \gamma_2 \in \mathrm{Iso}(\mathbb{R}^3, \mathrm{can})$ where

$$\gamma_1 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -x \\ -y \\ z \end{pmatrix} + \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}$$

and

$$\gamma_2 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ -y \\ -z \end{pmatrix} + \begin{pmatrix} 1/2 \\ 0 \\ 0 \end{pmatrix}.$$

for all $(x,y,z)^T \in \mathbb{R}^3$. Show that Γ is Bieberbach group and that $\Gamma \backslash \mathbb{R}^3$ is orientable.