Exercise Sheet 5

Exercise 1

Equip $\mathbb{R}^n$ with the Riemannian metric associated to the standard scalar product, i.e. for all $x \in \mathbb{R}^n$ and $v, w \in \mathrm{T}_x \mathbb{R}^n$ we set

$$g_x(v, w) = \langle v, w \rangle = \sum_{i=1}^{n} v_i w_i.$$

Show that the resulting Riemannian distance coincides with the Euclidean.

Solution sketch

Let $x, y \in \mathbb{R}^n$. Further, let $l \subseteq \mathbb{R}^n$ denote the straight line through x and y, and let $\mathrm{pr}_l : \mathbb{R}^n \to l$ denote the orthogonal projection onto l. Then $D_p \mathrm{pr}_l : \mathrm{T}_p \mathbb{R}^n \to \mathrm{T}_{\mathrm{pr}_l(p)} \mathbb{R}^n$ is distance non-increasing. Thus, if $c : [a, b] \to \mathbb{R}^n$ is the unit speed parameterization of the segment $\{x + t(y - x) \mid t \in [0, 1]\}$ and $\gamma : [a', b'] \to \mathbb{R}^n$ is any piecewise C^1-curve connecting x and y we have

$$l(\gamma) \geq l(\mathrm{pr}_l \circ \gamma) \geq l(c) = \|y - x\|$$

which is the Euclidean distance of x and y.

Exercise 2

Let $\Gamma \leq \mathrm{Iso}(\mathbb{R}^n, \mathrm{can})$ be a Bieberbach group consisting of pure translations. Show that there is a basis $(a_1, \ldots, a_n)$ of $\mathbb{R}^n$ such that

$$\Gamma = \{T_a \mid a \in \mathbb{Z}\, a_1 + \cdots + \mathbb{Z}\, a_n\}.$$

Solution sketch

Since Γ consists of pure translations, it may be regarded as a subgroup of $(\mathbb{R}^n, +)$. As such, it is discrete because it acts properly discontinuously: The neutral element $T_0 \in \Gamma$ must not be an accumulation point.

We now argue by induction: If $n = 1$, cocompactness of Γ implies $\Gamma \neq \{T_0\}$. By discreteness there is an element $a_1 \in \mathbb{R}$ of smallest absolute value such that $T_{a_1} \in \Gamma$. We then necessarily have $\Gamma = \langle T_a \rangle = \{T_a \mid a \in \mathbb{Z}\, a_1\}$.

By cocompactness, $\Gamma \neq \{T_0\}$. Pick $T_a \in \Gamma$ and $a_1 \in \mathbb{R}^n$ with $\Gamma \cap \langle a \rangle_{\mathbb{R}} = \mathbb{Z}\, a_1$, using that proper discontinuity of the action of Γ implies discreteness of Γ in $\mathbb{R}^n$. Let $\pi : \mathbb{R}^n \to \mathbb{R}^n \setminus \langle a \rangle_{\mathbb{R}}$ denote the canonical projection. Then $\pi(\Gamma)$ is again a Bieberbach group consisting of pure translations. Now proceed as before to produce a vector $a_2' \in \mathbb{R}^{n-1}$ and choose a lift $a_2 \in \pi^{-1}(a_2')$. Iterating this, where cocompactness implies that there is always a possible choice, ultimately produces a basis $(a_1, \ldots, a_n)$ of $\mathbb{R}^n$ such that $\Gamma = \{T_a \mid a \in \mathbb{Z}\, a_1 + \cdots + \mathbb{Z}\, a_n\}$.

Exercise 3

Let Γ be the subgroup of $\mathrm{Iso}(\mathbb{R}^3, \mathrm{can})$ generated by $\gamma_1, \gamma_2 \in \mathrm{Iso}(\mathbb{R}^3, \mathrm{can})$ where

$$\gamma_1 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -x \\ -y \\ z \end{pmatrix} + \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}$$

and

$$\gamma_2 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ -y \\ -z \end{pmatrix} + \begin{pmatrix} 1/2 \\ 0 \\ 0 \end{pmatrix}.$$

for all $(x, y, z)^T \in \mathbb{R}^3$. Show that Γ is a Bieberbach group and that $\Gamma \backslash \mathbb{R}^3$ is orientable.

Solution sketch

To see that Γ acts cocompactly on $\mathbb{R}^3$, note that Γ contains the purely translational subgroup

$$\langle \gamma_1^2, \gamma_2^2, (\gamma_1\gamma_2)^2 \rangle = \langle T_{e_3}, T_{e_1}, T_{e_2} \rangle.$$

and hence $\pi : C \subseteq \mathbb{R}^3 \to \Gamma \backslash \mathbb{R}^3$, where C denotes the unit cube is surjective.

To see that Γ acts freely on $\mathbb{R}^3$, we first record

$$\gamma_1^{-1} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -x \\ -y \\ z \end{pmatrix} + \begin{pmatrix} 1/2 \\ 1/2 \\ -1/2 \end{pmatrix} \quad \text{and} \quad \gamma_2^{-1} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ -y \\ -z \end{pmatrix} + \begin{pmatrix} -1/2 \\ 0 \\ 0 \end{pmatrix}.$$

Next, notice that $r(\Gamma) \leq \mathrm{O}(3)$ is given by $\langle r(\gamma_1), r(\gamma_2) \rangle \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Any element of Γ can be written as a word $w = w_1 \cdots w_n$ with $w_i \in \{\gamma_1, \gamma_1^{-1}, \gamma_2, \gamma_2^{-1}\}$. We argue that no such word has a fixed point, unless it represents the identity, distinguishing between the four possibilities for $r(w)$: If $r(w) = \mathrm{id}$ then w is a translation and hence does not have a fixed point unless it is the identity. Now consider the case $r(w) = r(\gamma_1)$. Then $w = (r(\gamma_1), t(w))$ for some $t(w) \in \mathbb{R}^3$. Thus $w(v) = v$, i.e. $r(\gamma_1)v + t(w) = v$, for some $v \in \mathbb{R}^3$ if and only if $t(w) = (2v_1, 2v_2, 0)$. However, one argues by induction on the length n of w that $t(w)_3 \neq 0$ if $r(w) = r(\gamma_1)$: If $n = 1$ then either $w = \gamma_1$ or $w = \gamma_1^{-1}$. In both cases, $t(w)_3 \neq 0$. Now consider a word $w = w_1 \cdots w_n$ of length n with $r(w) = r(\gamma_1)$. Note that the latter assumption implies that $|\{i \in \{1, \ldots, n\} \mid w_i \in \{\gamma_1, \gamma_1^{-1}\}|$ is odd and $\{j \in \{1, \ldots, n\} \mid w_j \in \{\gamma_2, \gamma_2^{-1}\}$ is even. Let $i \in \{1, \ldots, n\}$ be the smallest integer such that $w_i \in \{\gamma_1, \gamma_1^{-1}\}$. If $i > 1$ then the assumption applies to the word $w_i \cdots w_n$ and we have $w_1, \ldots, w_{i-1} \in \{\gamma_2, \gamma_2^{-1}\}$ which only change the sign of $t(w_i \cdots w_n)$ which is non-zero. If $i = 1$ then either i is the only integer in $\{1, \ldots, n\}$ for which $w_i \in \{\gamma_1, \gamma_1^{-1}\}$ or there are at least two more such integers. In the first case, w_1 changes the translational part $t(w_2 \cdots w_n)$ from zero to either $1/2$ or $-1/2$. In the second case, $r(w_2 \cdots w_n) = \mathrm{id}$, i.e. $w_2, \ldots, w_n$ is a translation. However, one argues by induction on the (even) word length of a translation that a translation has integer entries. Similar arguments for the other possible rotational parts.

Concerning proper discontinuity, argue by induction that $t(w)_i \in \mathbb{Z} \cup \frac{1}{2}\mathbb{Z}$ for all $w \in \Gamma$ and $i \in \{1, 2, 3\}$. Given $v \in \mathbb{R}^3$, let $U_v := C^3_{1/2}(v)$ be the open cube of side length 1 around v. Then there are only finitely many elements $w \in \Gamma$ for which $wU_v \cap U_v \neq \emptyset$.

As to orientability, note that the standard volume form $\omega := dx_1 \wedge dx_2 \wedge dx_3$ on $\mathbb{R}^3$ is invariant under γ_1 and γ_2, and hence under Γ. This reflects that γ_1 and γ_2 act orientation-preservingly on $\mathbb{R}^3$. Therefore ω descends to a nowhere-vanishing n-form on $\Gamma \backslash \mathbb{R}^3$ which shows that the latter is orientable.