Exercise Sheet 6

Exercise 1

Show that the defining formula for the Levi-Civita connection developed in Theorem 2.30 of the lecture does indeed yield a connection.

Solution sketch

We recall the defining formula of the Levi-Civita connection: Let M be a manifold and $X, Y, Z \in \Gamma(\mathrm{T}M)$. Then

$$\begin{aligned} 2g(\nabla_X Y, Z) = & Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \\ & + g([X,Y],Z) - g([X,Z],Y) - g([Y,Z],X). \end{aligned}$$

For $f \in C^\infty(M)$, one verifes $\nabla_{fX}Y = f\nabla_X Y$ and $\nabla_X fY = X(f)Y + f\nabla_X Y$ using various incarnations of the Leibniz rule such as

$$Yg(Z, fX) = Y(fg(Z,X)) = Y(f)g(Z,X) + fYg(Z,X)$$

and $g([fX,Y],Z) = g(-X(f)Y - f[X,Y],Z) = -X(f)g(Y,Z) - fg([X,Y],Z)$. The equality $\nabla_X Y - \nabla_Y X = [X,Y]$ is a consequence of cancellations.

Exercise 2

Let N be a smooth manifold and let $M \subseteq N$ be a regular submanifold of N. Show that every smooth vector field X on M extends to a smooth vector field on an open neighbourhood of M in N.

Solution sketch

Pick a locally finite cover (U_i, φ_i) of M in N with cubical chart codomains as in the definition of regular submanifold, and a partition of unity $(f_i)_{i \in I}$ on the open neighbourhood $\bigcup_{i \in I} U_i$ of M in N subordinate to the U_i. In each chart codomain, extend the push-forward of X defined on $\varphi_i(U_i \cap M)$ to the whole of $\varphi(U_i)$ by translations. Then pull back each extension to the respective X_i, multiply with f_i and sum over $i \in I$.

Exercise 3

Let M and N be smooth manifolds and let $f : M \to N$ be a smooth map. Show that

$$f^*\mathrm{T}N := \{(p,v) \in M \times \mathrm{T}N \mid v \in \mathrm{T}_{f(p)}N\}$$

can be equipped with the structure of a smooth vector bundle with base M.

Solution sketch

(This is an instance of the general notion of pullback bundle). Clearly, the map $\pi : f^*\mathrm{T}N \to M$, $(p,v) \mapsto p$ is smooth and surjective. Given local trivializations (U_i, h_i) of $\mathrm{T}N$, the pairs $(f^{-1}(U_i), \mathrm{id} \times (\mathrm{pr}_2 \circ h_i))$ are local trivializations of $f^*\mathrm{T}N$.