# Exercise Sheet 7

## Exercise 1

Show that the formula for the covariant derivative along a curve of the lecture satisfies the claimed properties (Theorem 2.37).

### Solution sketch

We recall the defining formula for the covariant derivative $D/dt$ along a curve $c : I \to M$: For $Y \in \Gamma(c^*\mathrm{T}M)$ and local coordinates $(U, \varphi)$ of $M$ at $c(t)$ with $\varphi(c(t)) = (x_1(t), \dots, x_m(t))$ we set

$$\frac{DY}{dt}(t) := \sum_{i=1}^{m} \left( Y_i'(t) + \sum_{j,k} \Gamma^i_{jk}(c(t)) x_j'(t) Y_k(t) \right) \partial_i(c(t)).$$

Given $f \in C^\infty(I)$, we compute

$$\begin{aligned}
\frac{D}{dt}(fY)(t) &= \sum_{i=1}^{m} \left( (fY_i)'(t) + \sum_{j,k} \Gamma^i_{jk}(c(t)) x_j'(t) f(t) Y_k(t) \right) \partial_i(c(t)) \\
&= \sum_{i=1}^{m} \left( f'Y_i(t) + fY_i'(t) + \sum_{j,k} \Gamma^i_{jk}(c(t)) x_j'(t) f(t) Y_k(t) \right) \partial_i(c(t)) \\
&= f'Y(t) + f\frac{DY}{dt}(t)
\end{aligned}$$

The assertion that $DY/dt(t) = (\nabla_{c'(t)}X)(c(t))$ whenever $X \in \Gamma(\mathrm{T}M)$ satisfies $Y(t) = X(c(t))$ for all $t \in I$ is already given in the proof presented in the lecture.

## Exercise 2

Prove the general formula on how the covariant derivative along a curve relates to the Riemannian metric (Proposition 2.42).

### Solution sketch

Given a Riemannian manifold $(M, g)$ with Levi-Civita connection $\nabla$ and a smooth curve $c : I \to M$, let $D/dt$ denote the associated covariant derivative along $c$. For $X, Y \in \Gamma(c^*\mathrm{T}M)$ we need to show that

$$\frac{d}{dt} g_{c(t)}(X(t), Y(t)) = g_{c(t)} \left( \frac{D}{dt} X(t), Y(t) \right) + g_{c(t)} \left( X(t), \frac{D}{dt} Y(t) \right).$$

To this end, expand $X = \sum_i X_i \partial_i$ and $Y = \sum_j Y_j \partial_j$ as usual to compute

$$\begin{aligned}
\frac{d}{dt} g_{c(t)}(X(t), Y(t)) &= \frac{d}{dt} \left( \sum_i \sum_j X_i(t) Y_j(t) g_{ij}(c(t)) \right) \\
&= \sum_i \sum_j (X_i'(t) Y_j(t) g_{ij}(c(t)) + X_i(t) Y_j'(t) g_{ij}(c(t)) + X_i(t) Y_j(t) g_{ij}'(c(t)))
\end{aligned}$$

Note that $g'_{ij}(t) = \sum_{k=1}^m x'_k(t)\partial_k g_{ij}(c(t))$. The term $g_{c(t)}\left(\frac{D}{dt}X(t), Y(t)\right)$ may is readily expanded using the definitions:

$$\begin{aligned} g_{c(t)}\left(\frac{D}{dt}X(t), Y(t)\right) &= g_{c(t)}\left(\sum_i (\dots)\,\partial_i(c(t)), \sum_l Y_l(t)\partial_l(c(t))\right) \\ &= \sum_i \sum_l \left(X'_i(t) + \sum_{j,k} \Gamma^i_{jk}(c(t))x'_j(t)X_k(t)\right) Y_l(t) g_{il}(c(t)) \end{aligned}$$

Similarly, we obtain $g_{c(t)}\left(X(t), \frac{D}{dt}Y(t)\right)$. Finally, match the terms using

$$2\sum_{l=1}^m \Gamma^l_{jk} g_{li} = \partial_j g_{ki} + \partial_k g_{ij} - \partial_i g_{jk}.$$