Exercise Sheet 8

Exercise 1

Let $\Gamma \leq \mathrm{Iso}(\mathbb{R}^n)$ be a Bieberbach group. Describe all geodesics on the Riemannian quotient manifold $\Gamma\backslash\,\mathbb{R}^n$.

Given a Riemannian manifold M, a geodesic $c_{(p,v)} : \mathbb{R} \to M$ is called periodic *if there is $t_0 > 0$ such that $c_{(p,v)}(t+t_0) = c_{(p,v)}(t)$ for all $t \in \mathbb{R}$.*

Describe all periodic geodesics on $\Gamma\backslash\,\mathbb{R}^n$ and compute their length.

Exercise 2

Let $K = \Gamma\backslash\,\mathbb{R}^2$ be the Klein bottle. Compute the parallel transport along the periodic geodesics obtained by projecting horizontal straight lines and vertical straight lines.

Exercise 3

Let $\mathbb{H}^2 = \{z \in \mathbb{C} \mid y \geq 0\}$ be the Poincaré upper half-plane with metric

$$\frac{(dx)^2 + (dy)^2}{y^2}.$$

(i) Show that the maps $\mathbb{R} \to \mathbb{H}^2$ given by $t \mapsto (x_0, \exp(at))$ are geodesics.

(ii) Show that every orientation-preserving isometry of $\mathbb{H}^2$ is given by

$$z \mapsto \frac{az+b}{cz+d} \quad \text{for some} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}).$$

(iii) Show that the $\mathrm{SL}(2,\mathbb{R})$-action on the unit tangent bundle

$$\mathrm{T}_1\,\mathbb{H}^2 = \{(z,v) \mid z \in \mathbb{H}^2,\ |v|_z = 1\}$$

is transitive.

(iv) Describe all geodesics of $\mathbb{H}^2$.

Exercise 4

Prove Corollary 2.53 of the lecture on how the exponential map connects points.