Exercise Sheet 8

Exercise 1

Let $\Gamma \leq \mathrm{Iso}(\mathbb{R}^n)$ be a Bieberbach group. Describe all geodesics on the Riemannian quotient manifold $\Gamma\backslash\mathbb{R}^n$.

Given a Riemannian manifold M, a geodesic $c_{(p,v)} : \mathbb{R} \to M$ is called periodic *if there is $t_0 > 0$ such that $c_{(p,v)}(t+t_0) = c_{(p,v)}(t)$ for all $t \in \mathbb{R}$.*

Describe all periodic geodesics on $\Gamma\backslash\mathbb{R}^n$ and compute their length.

Solution sketch

By Corollary 2.54 of the lecture, the geodesics on $\Gamma\backslash\mathbb{R}^n$ are exactly the images under $p : \mathbb{R}^n \to \Gamma\backslash\mathbb{R}^n$ of geodesics in $\mathbb{R}^n$. Let $c : \mathbb{R} \to \Gamma\backslash M$ be a periodic geodesic. If $\widetilde{c} : \mathbb{R} \to \mathbb{R}^n$ is a lift of c then $\widetilde{c}$ is a geodesic in $\mathbb{R}^n$ by Corollary 2.54 of the lecture. Also, since the Riemannian covering $p : \mathbb{R}^n \to \Gamma\backslash\mathbb{R}^n$ is a local isometry, it preserves the length of curves. Therefore, since $c(t+t_0) = c(t)$ for all $t \in \mathbb{R}$ and some $t_0 \in \mathbb{R}$, we conclude that the element $\gamma \in \Gamma = \pi_1(\Gamma\backslash M)$ determined by c acts as a translation on $\widetilde{c}(\mathbb{R})$. If $\widetilde{c}(t) = a + tb$ for some $a, b \in \mathbb{R}^n$ and $\gamma = (R, v)$ then $\gamma(\widetilde{c}(t)) = R(a+bt) + v = Ra + tRb + v \overset{!}{=} a + tb + t_0 b$. Therefore, the periodic geodesics of $\Gamma\backslash\mathbb{R}^n$ through $p(0)$ are in one-to-one correspondence with elements of Γ whose rotational part fixes a vector. The length is then given by the translational part.

Exercise 2

Let $K = \Gamma\backslash\mathbb{R}^2$ be the Klein bottle. Compute the parallel transport along the periodic geodesics obtained by projecting horizontal straight lines and vertical straight lines.

Solution sketch

See Gallot-Hulin-Lafontaine: Riemannian Manifolds, 3rd edition, 2.83b).

Exercise 3

Let $\mathbb{H}^2 = \{z \in \mathbb{C} \mid y \geq 0\}$ be the Poincaré upper half-plane with metric

$$\frac{(dx)^2 + (dy)^2}{y^2}.$$

(i) Show that the maps $\mathbb{R} \to \mathbb{H}^2$ given by $t \mapsto (x_0, \exp(at))$ are geodesics.

(ii) Show that every orientation-preserving isometry of $\mathbb{H}^2$ is given by

$$z \mapsto \frac{az+b}{cz+d} \quad \text{for some} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}).$$

(iii) Show that the $\mathrm{SL}(2,\mathbb{R})$-action on the unit tangent bundle

$$\mathrm{T}^1\,\mathbb{H}^2 = \{(z,v) \mid z \in \mathbb{H}^2,\ \|v\|_z = 1\}$$

is transitive.

(iv) Describe all geodesics of $\mathbb{H}^2$.

Solution sketch

For (i), recall from the lecture that with respect to the global chart $(\mathbb{H}^2, \mathrm{id})$, the Christoffel symbols are given by

$$\Gamma^1_{11}(z) = \Gamma^2_{12}(z) = \Gamma^1_{22}(z) = 0,\ \Gamma^2_{11}(z) = \frac{1}{y} \text{ and } \Gamma^1_{12}(z) = \Gamma^2_{22}(z) = -\frac{1}{y}$$

for every $z = x + iy \in \mathbb{H}^2$. Using these, one verifies that the maps $\gamma : \mathbb{R} \to \mathbb{H}^2$, $t \mapsto (x_0, \exp(at))$ are geodesics of $\mathbb{H}^2$ for every $x_0 \in \mathbb{R}$ and $a \in \mathbb{R}\setminus\{0\}$, i.e.

$$\gamma_i''(t) + \sum_{j,k=1}^{2} \Gamma^i_{jk}(\gamma(t))\gamma_k'(t)\gamma_j'(t) = 0$$

for $i \in \{1, 2\}$.

For part (iii), recall that $\mathrm{SL}(2, \mathbb{R})$ acts transitively on $\mathbb{H}^2$ and that we have $\mathrm{stab}_i(\mathrm{SL}(2, \mathbb{R})) = \mathrm{SO}(2)$. Denote

$$f_A : \mathbb{H}^2 \to \mathbb{H}^2,\ z \mapsto \frac{az + b}{cz + d} \quad \text{where} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R})$$

and compute

$$D_z f_A = \frac{ab - bc}{(cz + d)^2} = \frac{1}{(cz + d)^2}.$$

In particular, if

$$A := \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$$

we have

$$D_i f_A v = \frac{1}{(i\sin\varphi + \cos\varphi)^2} v = e^{-i2\varphi} v \quad (v \in \mathrm{T}_i\,\mathbb{H}^2 \cong \mathbb{C})$$

Hence $\mathrm{SO}(2)$ acts transitively on the tangent space $\mathrm{T}_i\,\mathbb{H}^2$. Combining this with the transitivity of $\mathrm{SL}(2, \mathbb{R})$ shows the asserted transitivity of $\mathrm{SL}(2, \mathbb{R})$ on $\mathrm{T}^1\,\mathbb{H}^2$.

The transitivity of $\mathrm{SL}(2, \mathbb{R})$ on $\mathrm{T}^1\,\mathbb{H}^2$ can be used to show that every isometry f of $\mathbb{H}^2$ which is orientation-preserving, is fractional linear: There is $T \in \mathrm{SL}(2, \mathbb{R})$ be such that $f_T \circ f(i) = i$ and $S \in \mathrm{SO}(2, \mathbb{R})$ such that $D_i f_S \circ f_T \circ f(1)) = 1$. Since f_S, f_T and f are orientation-preserving, this implies $D_i f_S \circ f_T \circ f = \mathrm{Id}$, which implies $f_S \circ f_T \circ f = \mathrm{id}$ whence f is fractional linear: We show that $f_S \circ f_T \circ f = \mathrm{id}$ fixes every point of $\mathbb{H}^2$: Since $\mathbb{H}^2$ is metrically complete with respect to the distance induced by the Riemannian metric, the Hopf-Rinow theorem implies that every point $q \in \mathbb{H}^2$ may be connected to $i \in \mathbb{H}^2$ via a geodesic γ_q. Since the image of γ_q is again a geodesic with the same initial data it has to coincide with γ_q, i.e. $f_S \circ f_T \circ f = \mathrm{id}$ fixey γ_q, in particular $q \in \mathbb{H}^2$.

Regarding (iv), the transitivity of $\mathrm{SL}(2, \mathbb{R})$ on $\mathrm{T}_1\,\mathrm{SL}(2, \mathbb{R})$ also implies that any geodesic is the image of the geodesic $\gamma : \mathbb{R} \to \mathbb{H}^2$, $t \mapsto (0, \exp(at))$ under an element of $\mathrm{SL}(2, \mathbb{R})$, whose shape is either a vertical line or a Euclidean half circle with center on the real axis.

Exercise 4

Prove Corollary 2.53 of the lecture on how the exponential map connects points.

Solution sketch

The construction of the pair (U, ε) has been clarified in the lecture in the meantime. The assertions (i), (ii) are then a consequence of Theorem 2.47 whereas (iii) is due to Proposition 2.52.