# Exercise Sheet 9

## Exercise 1

Prove Corollary 2.63 of the lecture, giving a criterion for a piecewise $C^1$-curve to be a geodesic.

### Solution sketch

If a piecewise $C^1$-curve $c : I \to M$, parametrized proportional to arc length, is locally length-minimizing in the sense that for every $t \in I$ there is $\varepsilon > 0$ such that $d(c(t-\varepsilon), c(t+\varepsilon)) = l(c([t-\varepsilon, t+\varepsilon]))$, then the proof of Proposition 2.66 implies that $c$ is also length-minimizing between any two points on $c$ in between $c(t-\varepsilon)$ and $c(t+\varepsilon)$. Therefore, considering totally normal neighbourhoods and Theorem 2.61 shows that $c$ is a geodesic. The converse is immediate from Theorem 2.61.

## Exercise 2

Show that $\mathrm{SL}(2,\mathbb{R})$ does not admit a bi-invariant metric.

### Solution sketch

Suppose that $\mathrm{SL}(2,\mathbb{R})$ does admit a bi-invariant metric. Then by a result of the lecture, its Lie group exponential coincides with the Riemannian exponential and is defined on the whole tangent space $\mathrm{T}_{\mathrm{Id}}\,\mathrm{SL}(2,\mathbb{R})$. From the Hopf-Rinow Theorem we conclude that it is surjective. It follows that any given element $g = \exp(X) \in \mathrm{SL}(2,\mathbb{R})$ is a square of some element in $\mathrm{SL}(2,\mathbb{R})$ since

$$g = \exp(X) = \exp\left(\frac{X}{2} + \frac{X}{2}\right) = \exp\left(\frac{X}{2}\right)^2.$$

However, one check by hand that for instance

$$\begin{pmatrix} -1 & 1 \\ & -1 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$$

is not a square of a matrix with real entries.

## Exercise 3

A Riemannian manifold $M$ is said to be *homogeneous* if for any two points $p, q \in M$ there is an isometry $f$ of $(M, g)$ with $f(p) = q$. Show that a homogeneous manifold is geodesically complete.

### Solution sketch

Homogeneity of the manifold implies that there is $\varepsilon > 0$ such that $\exp_p$ is defined on $B(0,\varepsilon) \subseteq \mathrm{T}_pM$ for all $p \in M$: Indeed, suppose that for a given $p \in M$, the exponential map $\exp_p$ is defined on $B(0,\varepsilon) \subseteq \mathrm{T}_pM$. Then given $q \in M$ and $f \in \mathrm{Iso}(M,g)$ with $f(p) = q$ the diagram

$$\begin{CD} M @>{f}>> M \\ @A{\exp_p}AA @AA{\exp_q}A \\ B(0,\varepsilon) \subseteq \mathrm{T}_pM @>{D_pf}>> B(0,\varepsilon) \subseteq \mathrm{T}_qM \end{CD}$$

commutes by uniqueness of geodesics and the fact that isometries preserve geodesics.

As a consequence, any geodesic defined on a closed interval $[a,b]$ can be extended to $(a-\varepsilon, b+\varepsilon)$. That is, $(M,g)$ is geodesically complete.