RIEMANNIAN GEOMETRY

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Abstract. These are notes of the course Differential Geometry II held at ETH Zurich in 2016.

Disclaimer. This is a preliminary version. Please report any typos, mistakes, comments etc. to stephan.tornier@math.ethz.ch.

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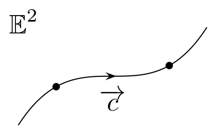
Introduction

In this introduction, we outline how Bernhard Riemann resolved three important problems of his time with the definition of what we now call a *Riemannian manifold* in his habilitation treatise [Rie54]. The three problems revolve around how to deal with the geometry of curves and surfaces after Gauss, Euclid's fifth postulate and manifolds which "can't be seen". This is a largely informal section. Its notation is deliberately old-fashioned and should not be taken too seriously.

Given a curve $\overrightarrow{c} : \mathbb{R} \to \mathbb{E}^2,\ t \mapsto \overrightarrow{c}(t)$ in Euclidean space and points $a, b \in \mathbb{R}$ with $a < b$ one can talk about the length of the segment $[\overrightarrow{c}(a), \overrightarrow{c}(b)]$:



$$\text{length}(\overrightarrow{c}[a,b]) = \int_a^b \|\overrightarrow{c}'(t)\| \, dt.$$

This defines the intrinsic metric of $\overrightarrow{c}$. Also, a curve $\overrightarrow{c}$ with $\overrightarrow{c}'(t) \neq 0$ for all $t \in \mathbb{R}$ can be reparametrized which is really saying that $\overrightarrow{c}(\mathbb{R})$ with its intrinsic metric is isomorphic to $\mathbb{R}$. This already finishes the metric classification of curves.

However, there are interesting invariants describing how the curve sits in Euclidean space such as curvature: A circle of radius r is defined to have constant curvature $1/r$. Given a curve $\overrightarrow{c} : \mathbb{R} \to \mathbb{E}^2$ and an orientation of $\overrightarrow{c}$ via normal vectors $\overrightarrow{n} : \mathbb{R} \to \mathbb{E}^2$ the curvature of $\overrightarrow{c}$ at the point $\overrightarrow{c}(t)$ is defined as

$$K(\overrightarrow{c}(t)) = \frac{\pm 1}{\text{radius of osculating circle at } \overrightarrow{c}(t)}$$

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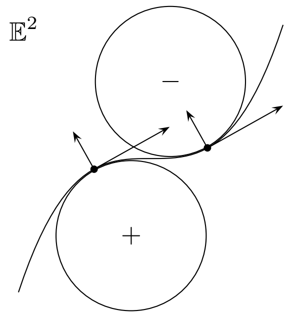
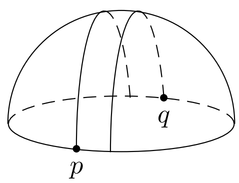
where the sign depends on whether $\overrightarrow{n}(t)$ points towards or away from the osculating circle which is the unique circle tangent of order three at $\overrightarrow{c}(t)$.



In other words, let $t_1 \leq t \leq t_2$. Then if $\overrightarrow{c}(t_1)$, $\overrightarrow{c}(t)$ and $\overrightarrow{c}(t_2)$ are not collinear, there is a unique circle through the three of them. Take the limit as t_1 and t_2 tend to t from below and above respectively.

Given a surface S in $\mathbb{E}^3$ and points p and q on S the intrinsic distance of p and q is defined as



$$d_S(p,q) = \inf \left\{ \text{length}(\overrightarrow{c}) \,\middle|\, \overrightarrow{c} : [0,1] \to \mathbb{E}^3 \text{ smooth} : \begin{cases} \overrightarrow{c}([0,1]) \subseteq S \\ \overrightarrow{c}(0) = p \\ \overrightarrow{c}(1) = q \end{cases} \right\}$$

One may then ask what the shortest path between p and q is. This question was studied at least since the work of the Bernoulli's on variational questions. The study of the distance on S and the determination of shortest paths − *geodesics* − amount to the intrinsic geometry of S.

A surface S as above, or rather a piece of it, can − being two-dimensional − be parametrized by two real parameters, say u and v. A curve $\overrightarrow{c}(t)$ lying on S can then be given by $\overrightarrow{c}(t) = (x(u(t),v(t)), y(u(t),v(t)), z(u(t),v(t)))^T$ and we compute

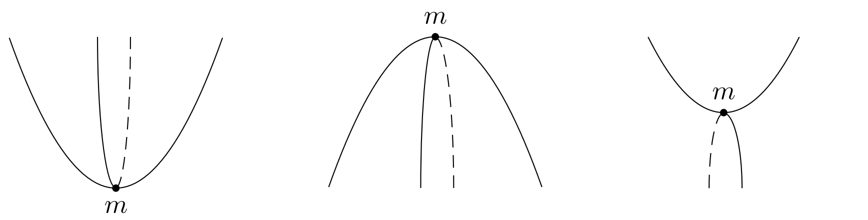
$$\|\overrightarrow{c}'(t)\|^2 = E\ u'(t)^2 + 2F\ u'(t)v'(t) + G\ v'(t)^2.$$

The expression $ds^2 = E\ du^2 + 2F\ dudv + G\ dv^2$ was used to refer to the *first fundamental form*. This notation should not be confused with notation from the theory of differential forms but rather be granted as moonshine from the 18th century. The point is that the inner distance on S determines and is determined by the first fundamental form.

What can be said about curvature in the case of surfaces? The mathematician's way to deal with it is to reduce the question to the case of curves discussed above. Let $m \in S$ and let $\overrightarrow{n}$ denote a normal vector to S at p. Further, let $\overrightarrow{u} \in \mathrm{T}_m S$ be a unit tangent vector and let P denote the plane spanned by $\overrightarrow{n}$ and $\overrightarrow{u}$. Then the intersection of S and P is a curve $\overrightarrow{c}$ and we define

$$\Pi : T_m S \to \mathbb{R},\ \overrightarrow{u} \mapsto K(\overrightarrow{c} \text{ at } m).$$

The map Π assumes both a minimum $K_1(m)$ and a maximum $K_2(m)$ which may take arbitrary signs. Gauss defined curvature to be the product of these two principal curvatures: $K(m) := K_1(m)K_2(m)$. This makes sense geometrically:



$K_1(m), K_2(m) > 0$ $\quad$ $K_1(m), K_2(m) < 0$ $\quad$ $K_1(m) < 0,\ K_2(m) > 0$

He proved the following *Theorema Egregium*.

Theorem (Gauss). The curvature K_S of a surface S only depends on the intrinsic geometry of S. It admits a formula in terms of the coefficients E, F and G of the first fundamental form.

He was also conscious of the fact that this notion of curvature assumes that *we can see* the surface under consideration inside Euclidean space $\mathbb{E}^3$ and hence does not lend itself to investigate e.g. projective space $\mathbb{P}^2(\mathbb{R})$ which is defined as

$$\mathbb{P}^2\,\mathbb{R} := \mathbb{R}^3 \setminus \{0\}/\sim$$

where $x \sim y$ if and only if there is $\lambda \in \mathbb{R} \setminus \{0\}$ such that $x = \lambda y$. The Klein bottle is another example.

We now turn to Euclid's fifth postulate, the parallel postulate. Euclid's approach to the geometry was to define the objects under consideration, namely points and lines, and to postulate "self-evident" statements, such as that there should be a unique line going through two distinct points, with the help of which further objects could be introduced and statements made. His fifth postulate stated that given a line l and a point P not on l there is a unique line through P not intersecting l.

In 1829, Lobatchevsky constructed a geometry that did not satisfy Euclid's fifth postulate. In fact Gauss had already known about this but had not published anything as he feared his reputation.

Finally, Bernhard Riemann introduced n-dimensional manifolds and their tangent spaces in his habilitation treatise in 1854. He suggested to smoothly put a scalar product $\langle -, - \rangle_m$ on each tangent space $\mathrm{T}_m M$ of a manifold of M which allows one to define the notion of length of a curve in M and geodesics. He also introduced what is now called the *Riemann curvature tensor*, a revolutionary way to measure curvature. It is related to sectional curvature, Ricci curvature and scalar curvature. Sectional curvature is a curvature notion that generalizes Gauss curvature: Let $P \subseteq \mathrm{T}_m M$ be a two-dimensional vector subspace. Given $\varepsilon > 0$, consider the "circle" $C_P(m, \varepsilon)$ of points in M that are reached from m by following the geodesic flow along unit tangent vectors in P. Then the length of the closed curve $C_P(m, \varepsilon)$ is given by

$$\operatorname{length}(C_P(m, \varepsilon)) = 2\pi\varepsilon - \frac{\pi}{3}\sigma_m(P)\varepsilon^3 + \mathcal{O}(\varepsilon^4)$$

where $\sigma_m(P)$ is the sectional curvature which, as a consequence, is an intrinsic object. By the way, note that it is a big mystery why there never is a term in ε^2.

Riemann ended his habilitation treatise with the example of the open unit disk $\{x \in \mathbb{R}^n \mid \sum_{i=1}^n x_i^2 < 1\}$ in $\mathbb{R}^n$ on which he defines the metric

$$ds^2 = \frac{dx_1^2 + \cdots + dx_n^2}{\left(1 - \sum_{i=1}^n x_i^2\right)^2}.$$

This coincides exactly with Lobatchevsky's non-Euclidean plane. Overall, Riemann's habilitation treatise dealt with many problems in one stroke.

Course Outline. In this course, we will define Riemannian metrics on smooth manifolds and use them to study geodesics. We also study derivates of vector fields with respect to each other, leading to the notion of *connection*. In general, there are many possible connections. However, on a Riemannian manifold there is a preferred one, the *Levi-Civita connection*. Using the framework of connections we will extremely efficiently identify geodesics as curves whose acceleration vanishes. After introducing various notions of curvature we move on to the relation between local curvature properties of a Riemannian manifold and its global properties, e.g. properties of its fundamental group or de Rham cohomology spaces. For instance, there is the following classical theorem.

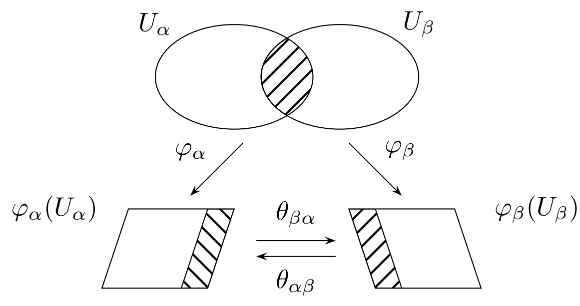
Theorem (Berger-Klingenberg). Let M be a simply-connected smooth manifold and assume that $1/4 < \sigma_m(P) \leq 1$ for all $m \in M$ and $P \leq \mathrm{T}_m M$ be two-dimensional. Then M is homeomorphic to S^n where $n = \dim M$.

In the context of dynamical systems and ergodic theory we shall see that the negative curvature world is very different from the above.

1. Review of Differential Geometry

In this chapter, we recall some fundamentals from differential geometry. Whenever possible, we refer to the lecture notes [BT15] of the Differential Geometry I course of the fall semester 2015. The main new topics include a more thorough treatment of vector fields, basic concepts of vector bundles, a short section on Lie groups and one on covering maps and fibrations.

1.1. **Smooth Manifolds and Smooth Maps.** Recall that a *topological manifold of dimension n* is a topological space M which is Hausdorff, second-countable and locally homeomorphic to $\mathbb{R}^n$. A *smooth atlas* $\mathcal{A}$ on M is a collection of charts $\mathcal{A} = \{(U_\alpha, \varphi_\alpha) \mid \alpha \in A\}$ such that $\bigcup_{\alpha \in A} U_\alpha = M$ and all coordinate transformations $\theta_{\beta\alpha} = \varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ is smooth.



One shows that every smooth atlas is contained in a unique *maximal* one. A *smooth manifold of dimension n* is a topological n-manifold together with a maximal smooth atlas. Recall that an exponent of a manifold typically indicates its dimension. Let M^m and N^n be smooth manifolds and let $f : M \to N$ be a map. We say that f is *differentiable at* $p \in M$ if there are charts (U, φ) at $p \in M$ and (V, ψ) at $f(p)$ such that $f(U) \subseteq V$ and $\psi f \varphi^{-1} : \varphi(U) \to \psi(V)$ is differentiable at $\varphi(p)$. Further, f is *smooth* if it is continuous and if for all charts (U, φ) of M and (V, ψ) of N satisfying $f(U) \subseteq V$ the map $\varphi(U) \to \psi(V)$ is smooth. Note that the continuity assumption in the last definition is crucial as a map that is wild to the extent that there are no pairs of charts as above should not be termed smooth. Given a map $f : M \to N$ that is differentiable at $p \in M$ the *rank* $\mathit{rank}_p(f)$ *of* f *at*

p is defined to be the rank of the linear map $D_p f : \mathrm{T}_p M \to \mathrm{T}_{f(p)} N$. We further recall that a smooth map $f : M^m \to N^n$ is an *immersion* if $\mathrm{rank}_p(f) = m$ for all $p \in M$, an *submersion* if $\mathrm{rank}_p(f) = n$ for all $p \in M$ and an *embedding* if it is an immersion and a homeomorphism onto its image. In the last situation, $f(N)$ is a regular submanifold and f is a diffeomorphism onto its image.

An important tool to construct manifolds is the following.

Theorem 1.1. Let M^m and N^n be smooth manifolds and let $f : M^m \to N^n$ be a smooth map of constant rank k. Then for any $y \in f(M)$ the subset $f^{-1}(y) \subseteq M$ is a regular submanifold of dimension $m - k$.

By applying Theorem 1.1 in the case where M and N are vector spaces and f is a linear map, note that it is a non-linear version of the first isomorphism theorem from linear algebra.

Example 1.2. (Orthogonal Groups). Let $q : \mathbb{R}^n \to \mathbb{R}$ be a quadratic form. Then there is a matrix $A \in \mathrm{Sym}(n, \mathbb{R})$ such that $q(x) = x^T Ax$. Recall Silvester's classification of quadratic forms via their *signature* and assume from now on that q is non-degenerate, equivalently, that $A \in \mathrm{GL}(n, \mathbb{R})$. For example, if $n = 2$ we may have $q(x_1, x_2)^T = x_1^2 - x_2^2$. Notice that q producing vectors of length zero says nothing about it being degenerate or non-degenerate. The *orthogonal group* of q is

$$\begin{aligned} \mathrm{O}(q) &:= \{h \in \mathrm{GL}(n, \mathbb{R}) \mid q(hx) = q(x)\ \forall x \in \mathbb{R}^n\} \\ &= \{h \in \mathrm{GL}(n, \mathbb{R}) \mid h^T Ah = A\} \end{aligned}$$

Also recall that $\mathrm{O}(q)$ is compact if and only if q is either positive or negative definite. Regardless of the signature of q, as long as it is non-degenerate, $\mathrm{O}(q)$ has dimension $n(n-1)/2$ as can be seen using Theorem 1.1: Namely, consider the map

$$f : \mathrm{GL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R}),\ g \mapsto g^T Ag.$$

In order to compute the derivative of f we recall that $\mathrm{GL}(n, \mathbb{R})$ acquires its manifold structure as an open subset of $M_{n,n}(\mathbb{R}) \cong \mathbb{R}^{n \cdot n}$. As such we identify $\mathrm{T}_g \mathrm{GL}(n, \mathbb{R})$ with $\mathrm{M}_{n,n}(\mathbb{R})$ for every $g \in \mathrm{GL}(n, \mathbb{R})$. We compute

$$(D_g f)(Y) = Y^T Ag + g^T AY$$

for all $Y \in \mathrm{T}_g \mathrm{GL}(n, \mathbb{R})$. Observe, that the map $D_g f$ takes values in $\mathrm{Sym}(n, \mathbb{R})$. In fact every $S \in \mathrm{Sym}(n, \mathbb{R})$ is in the image of $D_g f$ as

$$D_g f \left(\frac{1}{2} A^{-1} (g^{-1})^T S \right) = S.$$

Consequently, $\mathrm{rank}(D_g f) = n(n+1)/2$ and the constant rank theorem implies that $\mathrm{O}(q) = f^{-1}(A)$ is a submanifold of dimension $n^2 - n(n+1)/2 = n(n-1)/2$.

The case in which q is degenerate is left to the reader.

Example 1.3. (Symplectic Group). Let V be a finite-dimensional real vector space and let $\omega : V \times V \to \mathbb{R}$ be an alternating bilinear 2-form. If ω is non-degenerate, i.e. $\{x \in V \mid \omega(x, y) = \{0\}\ \forall y \in V\} = 0$, it is called *symplectic*. In this case, $\dim V = 2n$ and there is a basis $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ such that $\omega(x, y) = x^T Jy$ where

$$J = \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix}$$

Note the sharp contrast to the variety of quadratic forms. The *symplectic group* is given by

$$\begin{aligned} \mathrm{Sp}(2n, \mathbb{R}) &= \{g \in \mathrm{GL}(2n, \mathbb{R}) \mid \omega(gx, gy) = \omega(x, y)\ \forall x, y \in \mathbb{R}^{2n}\} \\ &= \{g \in \mathrm{GL}(2n, \mathbb{R}) \mid g^T Jg = J\}. \end{aligned}$$

It is an exercise to show that $\mathrm{Sp}(2n,\mathbb{R})$ is a submanifold of $\mathrm{GL}(n,\mathbb{R})$ of dimension $n(2n+1)$.

We shall see later that admitting a a symplectic 2-form is an interesting invariant for manifolds, in contrast to being Riemannian.

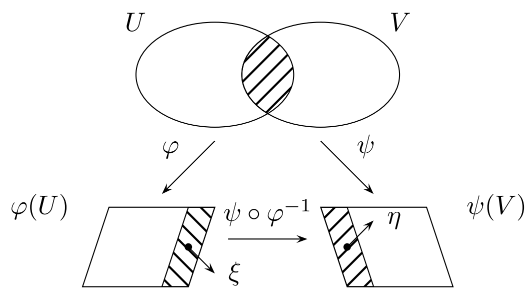
1.2. **Tangent Bundle.** Given a smooth manifold M^m and $p \in M$, the *tangent space* T_pM *of* M *at* p is defined as the quotient of

$$\mathcal{A}_p := \{(U,\varphi,\xi) \mid (U,\varphi) \text{ is a chart at p}, \ \xi \in \mathbb{R}^n\}$$

by the equivalence relation

$$(U,\varphi,\xi) \sim_p (V,\psi,\eta) \ \Leftrightarrow D_{\varphi(p)}(\psi\varphi^{-1})\xi = \eta.$$

In this way, $\mathrm{T}_pM := \mathcal{A}_p \,/ \sim_p$ is an m-dimensional vector space canonically attached to a point $p \in M$.



Note that every chart (U,φ) of M at p gives rise to a vector space isomorphism $\theta_{(U,\varphi,p)} : \mathbb{R}^m \to \mathrm{T}_pM$ by setting $\xi \mapsto [(U,\varphi,\xi)]$.

If $f : M \to N$ is a map which is differentiable at $p \in M$ then there is a canonical linear map $D_pf : \mathrm{T}_pM \to \mathrm{T}_{f(p)}N$ such that whenever (U,φ) is a chart at p and (V,ψ) is a chart at $f(p)$ with $f(U) \subseteq V$, the diagram

$$\begin{array}{ccc} \mathbb{R}^m & \xrightarrow{\theta_{(U,\varphi,p)}} & \mathrm{T}_pM \\ \downarrow{\scriptstyle D_{\varphi(p)}\psi f \varphi^{-1}} & & \downarrow{\scriptstyle D_pf} \\ \mathbb{R}^m & \xrightarrow[\theta_{(V,\psi,f(p))}]{} & \mathrm{T}_{f(p)}N \end{array}$$

commutes. There is an alternative, more classical and convenient way to define tangent vectors. It fits into the paradigm of reducing new problems to the case of curves. Consider the set C_p of pairs (I,c) consisting of an open interval $I \subseteq \mathbb{R}$ containing $0 \in \mathbb{R}$ and smooth curve $c : I \to M$ with $c(0) = p$. We declare $(I,c) \sim_p (J,\gamma)$ if there is a chart (U,φ) at p such that $(\varphi \circ c)'(0) = (\varphi \circ \gamma)'(0)$.

Lemma 1.4. Let M be a smooth manifold and let $p \in M$. Further, let (U,φ) be a chart of M at p. Then the map

$$C_p \to \mathcal{A}_p, \ (I,c) \mapsto (U,\varphi,(\varphi \circ c)'(0))$$

induces a well-defined bijection $C_p/\sim_p \to \mathcal{A}_p \,/ \sim_p$.

The verification of Lemma 1.4 is left to the reader.

1.2.1. *Tangent bundle.* Finally, we now turn to the *tangent bundle*, see [BT15, Sec. 2.3]. As a set, the tangent bundle of a smooth manifold M is defined as

$$\mathrm{TM} := \bigcup_{x\in M} \{x\} \times \mathrm{T}_x M = \{(x,v) \mid x \in M,\ v \in \mathrm{T}_x M\}.$$

It comes with the canonical projection map $\pi : \mathrm{TM} \to M,\ (x, v) \mapsto x$. Given an open set $U \subseteq M$ we set $\mathrm{T}U = \pi^{-1}(U)$. Further, if (U, φ) is a chart of M, we define the map $D\varphi : \mathrm{T}U \to \varphi(U) \times \mathbb{R}^n$ by $(x, v) \mapsto (\varphi(x), (D_x\varphi)v)$.

Lemma 1.5. Let M^m be a smooth manifold. Then $\mathrm{T}M$ admits a topological $2m$-manifold structure for which $\{(\mathrm{T}U, D\varphi) \mid (U, \varphi) \text{ chart}\}$ is a smooth atlas.

1.2.2. *Vector bundles.* The tangent bundle is an example of the more general notion of *vector bundle* which we introduce now.

Definition 1.6. A *vector bundle* is a triple (π, E, B) consisting of a smooth map $\pi : E \to B$ of smooth manifolds such that

(i) π is surjective,
(ii) there is an open cover $(U_i)_{i\in I}$ of B and a collection of diffeomorphisms h_i $(i \in I)$
$$h_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^n$$
such that $h_i(\pi^{-1}(x)) = \{x\} \times \mathbb{R}^n$, and
(iii) for all $i, j \in I$, the map $h_i(h_j|_{U_i\cap U_j})^{-1} : (U_i \cap U_j) \times \mathbb{R}^n \to (U_i \cap U_j) \times \mathbb{R}^n$ given by $(x, v) \mapsto (x, g_{ij}(x)v)$, where $g_{ij} : U_i \cap U_j \to \mathrm{GL}(n, \mathbb{R})$, is smooth.

It is an exercise to show that the tangent bundle of a manifold is in fact a vector bundle in the sense of Definition 1.6. Part (iii) of said definition entails that each fiber $\pi^{-1}(x)$ $(x \in B)$ has a well-defined vector space structure.

Example 1.7. As another example of vector bundles, recall the Grassmannian $G(k, n) := \{L \subseteq \mathbb{R}^n \mid \dim L = k\}$ and define

$$E(k, n) = \{(L, v) \mid L \in G(k, n),\ v \in L\}$$

In a sense to be made precise, these vector bundles are universal.

Definition 1.8. Let (π, E, B) be a vector bundle.

(i) A (smooth) *section* of (π, E, B) is a (smooth) map $s : B \to E$ such that $\pi \circ s = \mathrm{id}_B$.
(ii) The vector bundle (π, E, B) is *trivial* if there is a diffeomorphism $h : E \to B\times\mathbb{R}^n$ such that $h|_{\pi^{-1}(x)} : \pi^{-1}(x) \to \{x\}\times\mathbb{R}^n$ is a vector space isomorphism for every $x \in B$.

Note that for any vector bundle, the zero section is smooth.

1.3. **Vector Fields.**

Definition 1.9. Let M be a smooth manifold. A (smooth) *vector field* on M is a (smooth) section $X : M \to \mathrm{T}M$ of the tangent bundle.

Let $\Gamma(\mathrm{T}M)$ denote the span of all smooth vector fields on M. This space is not only a real vector space but in fact a module over $C^\infty(M)$: Given $f \in C^\infty(M)$ and $X \in \Gamma(\mathrm{T}M)$ we define $(fX)(x) := f(x)X(x)$ for all $x \in M$.

Proposition 1.10. Let M be a smooth manifold. Then $\mathrm{T}M$ is trivial if and only if there are smooth vector fields $X_1, \ldots, X_m$ on M such that $(X_1(x), \ldots, X_m(x))$ is a basis of $\mathrm{T}_x M$ for all $x \in M$.

A manifold is *parallelizable* if its tangent bundle is trivial. When mathematicians first looked at this notion they were surprised to find that there are non-parallelizable manifolds, for instance almost all spheres.

Example 1.11.

(i) It is evident that $\mathbb{R}^n$ is parallelizable for all n.

(ii) We now show that also all tori $T^n := (S^1)^n$ are parallelizable. To this end, we view S^1 as the abelian group $S^1 := \{z \in \mathbb{C} \mid |z| = 1\}$. Recall that for every $\varrho \in T^n$ the left multiplication map $L_\varrho : T^n \to T^n,\ z \mapsto \varrho z$ is a diffeomorphism. A trivialization of $\mathrm{T}(T^n)$ is thus given by

$$\mathrm{T}(T^n) \to T^n \times \mathrm{T}_{\mathbb{1}}(T^n),\ (\varrho, v) \mapsto (\varrho, D_\varrho L_{\varrho^{-1}}(v))$$

This argument applies in the more general context of Lie groups.

(iii) As to the spheres, it is a fact that S^n is parallelizable if and only if $n \in \{1, 3, 7\}$. In these cases, the respective sphere can be identified as the unit length elements in a real division algebra, namely either the complex numbers, the quaternions or the octonions. The case of S^1 is covered above.

The key parallelizability does not lie in a manifold's fundamental group but in its higher-dimensional fundamental groups. Given Proposition 1.10, the following theorem shows that all even-dimensional spheres, except S^0, are not parallelizable. The proof we give is due to Milnor and is not very telling but nicely uses the theory developed in part one of this course.

Theorem 1.12. The sphere S^n admits an everywhere non-zero smooth vector field if and only if n is odd.

Proof. We first show that the condition is sufficient: If $n = 2m-1$ we may construct a nowhere vanishing vector field as follows: Exhibit

$$S^{2m-1} = \left\{ (x_1, y_1), \dots, (x_m, y_m) \,\middle|\, \sum_{i=1}^{m} (x_i^2 + y_i^2) = 1 \right\} \subseteq \mathbb{R}^{2m}.$$

Now, given the rotation

$$r(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \in \mathrm{SO}(2)$$

define $R(t) := r(t) \oplus \cdots \oplus r(t)$ acting on $\mathbb{R}^{2m}$. Then $R(t)(S^{2m-1}) \subseteq S^{2m-1}$ and we may set

$$X(z) := \left.\frac{d}{dt}\right|_{t=0} \{t \mapsto R(t)z\}$$

for all $z \in S^{2m-1}$. One verifies that $\langle X(z), z\rangle = 0$ and $\|X(z)\| = 1$. Therefore, X is a unit vector field on S^{2m-1}.

To show necessity we argue by contradiction. Assume that there is a smooth map $X : S^n \to \mathbb{R}^{n+1}$ such that $\langle X(x), x\rangle = 0$ for all $x \in S^n$. Multiplying with an appropriate smooth function we may assume that $\|X(x)\| = 1$ for all $x \in S^n$. Now, let $\varepsilon > 0$ and define $f_\varepsilon : S^n \to \mathbb{R}^{n+1}$ by $f_\varepsilon(x) := x + \varepsilon X(x)$. Then $f_\varepsilon(x)$ has norm $\sqrt{1+\varepsilon^2}$ for every $x \in X$ since

$$\langle f_\varepsilon(x), f_\varepsilon(x)\rangle = \langle x + \varepsilon X(x), x + \varepsilon X(x)\rangle = \langle x, x\rangle + \varepsilon^2 \langle X(x), X(x)\rangle = 1 + \varepsilon^2.$$

Overall, f_ε is a smooth map taking values in $S^n(\sqrt{1+\varepsilon^2})$. In order to be able to talk about the degree of f_ε we endow $S^n(r)$ $(r > 0)$ with the orientation it obtains as boundary of the regular domain $B^{n+1}(r) = \{x \in \mathbb{R}^{n+1} \mid \|x\| \le r\}$.

In fact, $\deg f_\varepsilon = 1$ which can be seen as follows: Consider the map projection $M_\varepsilon : S^n(\sqrt{1+\varepsilon^2}) \to S^n$ given by $y \mapsto y/\|y\|$ which is an orientation-preserving diffeomorphism. Precomposing it with f_ε yields $M_\varepsilon \circ f_\varepsilon(x) = (x+\varepsilon X(x))/(\sqrt{1+\varepsilon^2})$ which is homotopic to $M_0 f_0 = \mathrm{Id}_{S^n}$. Hence $\deg M_\varepsilon \circ f_\varepsilon = 1$ and thus $\deg f_\varepsilon = 1$.

To obtain the announced contradiction, consider the differential form

$$\omega = \sum_{i=1}^{n+1} (-1)^i x_i \; x_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_{n+1} \in \Omega^n(\mathbb{R}^{n+1}) \in \Omega^n(\mathbb{R}^{n+1}).$$

For instance, if $n = 3$, then $\omega = x_1 \; dx_2 \wedge dx_3 - x_2 \; dx_1 \wedge dx_3 + x_3 \; dx_1 \wedge dx_2$. Observe

$$\begin{aligned} d(x_i \; dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_{n+1}) &= dx_i \wedge dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_{n+1} \\ &= (-1)^{i-1} \; dx_1 \wedge \cdots \wedge dx_{n+1}. \end{aligned}$$

Hence $d\omega = (n+1) dx_1 \wedge \cdots \wedge dx_{n+1}$. Let $\omega|_{S^n(r)}$ denote the pullback of ω via the injection $S^n(r) \to \mathbb{R}^{n+1}$. On the one hand, Stokes' Theorem yields

$$\int_{S^n(r)} \omega|_{S^n(r)} = \int_{B^{n+1}(r)} d\omega = (n+1) \operatorname{vol}(B^{n+1}(r)) = (n+1) c_n r^{n+1}$$

where c_n is a constant depending solely on n. In particular, for $r = \sqrt{1+\varepsilon^2}$ we get

$$\int_{S^n(r)} \omega|_{S^n(r)} = (n+1) c_n (1+\varepsilon^2)^{\frac{n+1}{2}}.$$

On the other hand, we have

$$\int_{S^n(1)} (f_\varepsilon)^*(\omega|_{S^n(r)}) = \deg f_\varepsilon \int_{S^n(r)} \omega|_{S^n(r)} = \int_{S^n(r)} \omega|_{S^n(r)} = (n+1) c_n (1+\varepsilon^2)^{\frac{n+1}{2}}.$$

Regarding the leftmost expression, note that $f_\varepsilon^*(\omega|_{S^n(r)})$ is a differential form on $S^n(1)$ which depends on ε. By definition

$$(f_\varepsilon^* \omega)_x(v_1, \ldots, v_n) = \omega_{f_\varepsilon(x)}(D_x f_\varepsilon v_1, \ldots, D_x f_\varepsilon v_n)$$

which is a sum over an index i ranging between 1 and $n+1$ in which the i-th summand is of the form

$$(x_i + \varepsilon X_i(x))(dx_1 \wedge \cdots \widehat{dx_i} \wedge \cdots \wedge dx_{n+1})((\mathrm{Id} + \varepsilon D_x X) v_1, \ldots, (\mathrm{Id} + \varepsilon D_x X) v_{n+1})$$

On this expression, one sees that $(f_\varepsilon^* \omega|_{S^n(r)}) = \sum_{j=0}^n \varepsilon^j \eta_j$ where $\eta_j \in \Omega^n(S^n)$. We obtain

$$\int_{S^n(1)} (f_\varepsilon^*)(\omega|_{S^n(r)}) = \sum_{j=0}^n \varepsilon^j \int_{S^n(1)} \eta_j$$

which is a polynomial in ε. On the other hand, if n is even then $(1+\varepsilon^2)^{(n+1)/2}$ is not a polynomial. □

There is a shorter proof of Theorem 1.12, see e.g. [Lee10, Thm. 13.32], using a nowhere-vanishing vector field to build a homotopy between the identity and the antipodal map.

1.3.1. *Vector fields and derivations.* Given a manifold M and a point $p \in M$, recall that a *derivation* at p is a linear map $\lambda : C^\infty(M) \to \mathbb{R}$ which satisfies $\lambda(fg) = f(p)\lambda(g) + g(p)\lambda(f)$. Let $\mathrm{Der}_p(C^\infty(M))$ denote the space of all derivations at $p \in M$. We have shown the following.

Proposition 1.13. Retain the above notation. The map

$$\mathrm{T}_p M \to \mathrm{Der}_p(C^\infty(M)), \; v \mapsto \{f \mapsto (D_p f)(v)\}$$

is a vector space isomorphism.

In what is to follow we will often abuse notation by identifying a tangent vector with a derivation and vice-versa.

In order to express vector fields in local coordinates, let (U, φ) be a chart of M and $p \in U$. For $f \in C^\infty(M)$, define

$$\partial_j(p)(f) := \frac{\partial(f \circ \varphi^{-1})}{\partial x_j}(\varphi(p)).$$

Then $\partial_j(p) \in \mathrm{Der}_p C^\infty(M)$ and $(\partial_1(p), \ldots, \partial_m(p))$ is a basis of $\mathrm{T}_p M \cong \mathrm{Der}_p C^\infty(M)$. Therefore, any vector field $X : M \to \mathrm{T}M$ can be expressed as

$$X(p) = \sum_{i=1}^m X_i(p)\partial_i(p)$$

on U with well-defined functions $X_i : U \to \mathbb{R}$. In this context, the following lemma is an important exercise.

Lemma 1.14. Retain the above notation. The vector field X is smooth if and only if its local expression in every chart is given by smooth functions.

We now exhibit a hidden algebra structure on $\Gamma(\mathrm{T}M)$ to which end we recall the following definition: A *derivation* of $C^\infty(M)$ is a linear map $\delta : C^\infty(M) \to C^\infty(M)$ such that $\delta(fg) = \delta(f)g + f\delta(g)$ for all $f, g \in C^\infty(M)$. Derivations can be defined for any associative algebra over any field, they are purely algebraic objects.

Now, given a smooth vector field $X \in \Gamma(\mathrm{T}M)$ on M and $f \in C^\infty(M)$ we define $L_X(f)(p) = X(p)f$. Then the map $L_X : C^\infty(M) \to C^\infty(M)$ is a well-defined linear map. Furthermore, we have the following.

Proposition 1.15. Retain the above notation. The map $L_X : C^\infty(M) \to C^\infty(M)$ is a derivation and the map $\Gamma(\mathrm{T}M) \to \mathrm{Der}(C^\infty(M))$, $X \mapsto L_X$ is an isomorphism.

Proof. The fact that L_X is a derivation follows from the fact that $f \mapsto (L_X f)(p)$ is a derivation at $p \in M$.

Injectivity of the map $\Gamma(\mathrm{T}M) \to \mathrm{Der}(C^\infty(M))$, $X \mapsto L_X$ follows from the fact that $\mathrm{T}_p M$ and $\mathrm{Der}_p(C^\infty(M))$ are isomorphic.

We now turn to surjectivity: Let $\delta \in \mathrm{Der}(C^\infty(M))$. Then for every $p \in M$, the map $f \mapsto \delta(f)(p)$ is a derivation at p. Hence there is a well-defined vector $X(p) \in \mathrm{T}_p M$ such that $\delta(f)(p) = X(p)(f)$ for all $f \in C^\infty(M)$. Together, these vectors form a vector field X on M. We need to verify that X is smooth. To this end, we argue in local coordinates: Let (U, φ) be a chart of M and let $X(p) = \sum_{i=1}^m X_i(p)\, \partial_i(p)$ $(p \in U)$ be the associated representation of X. Smoothness of X amounts to smoothness of the coefficient functions X_i $(i \in \{1, \ldots, m\})$. For every $f \in C^\infty(M)$, the map $p \mapsto \delta(f)(p) = X(p)f$ is smooth. In particular, the map $p \mapsto \sum_{i=1}^m X_i(p)\partial_i(p)(f)$ is smooth on U. Now recall that

$$\partial_i(p)(f) = \frac{\partial(f \circ \varphi^{-1})}{\partial x_i}(\varphi(p)).$$

Hence, if $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}$ was given by $x \mapsto x_l$ we would obtain $\partial_i(p)(f_l) = \delta_{il}$ and conclude that X_l is smooth. However, there may not be such a function defined on the whole of M. Nevertheless, we can remedy the argument by multiplying with a smooth bump function. □

The interpretation of smooth vector fields on a manifold M as derivations of $C^\infty(M)$ is important because it uncovers an important structure on $\Gamma(\mathrm{T}M)$, namely a Lie algebra structure.

Definition 1.16. Let W be a vector space and $A, B \in \mathrm{End}(W)$. The *bracket* of A and B is given by $[A, B] := AB - BA \in \mathrm{End}(W)$.

The relevance of this notion in our context is due to the following.

Lemma 1.17. Let M be a manifold and let $\delta, \delta' \in \mathrm{Der}(C^\infty(M))$. Then $[\delta, \delta'] \in \mathrm{Der}(C^\infty(M))$.

Note that one would not expect a composition of derivations to be a derivation in general as second-order derivatives might be introduced.

Proof. (Lemma 1.17). We compute

$$\delta\delta'(fg) = \delta(\delta'(f)g + f\delta'(g)) = \delta\delta'(f)g + \delta'(f)\delta(g) + \delta(f)\delta'(g) + \delta\delta'(g)f$$

as well as

$$\delta'\delta(fg) = \delta'(\delta(f)g + f\delta(g)) = \delta'\delta(f)g + \delta(f)\delta'(g) + \delta'(f)\delta(g) + \delta'\delta(g)f$$

and therefore

$$[\delta, \delta'](fg) = [\delta, \delta'](f)g + [\delta, \delta'](g)f.$$

□

Definition 1.18. Let M be a smooth manifold and let X, Y be smooth vector fields on M. The bracket $[X, Y]$ of smooth vector fields X and Y is the smooth vector field corresponding to $[L_X, L_Y] \in \mathrm{Der}(C^\infty(M))$ via the isomorphism $\Gamma(\mathrm{T}M) \to \mathrm{Der}(C^\infty(M))$.

Symbolically, we have $L_{[X,Y]} = [L_X, L_Y]$. We now compute $[X, Y]$ in local coordinates. Let (U, φ) be a chart of M and write

$$X(p) = \sum_{i=1}^m X_i(p)\partial_i(p) \quad \text{and} \quad Y(q) = \sum_{j=1}^m Y_j(q)\partial_j(q)$$

for $p, q \in U$. Then for every $f \in C^\infty(M)$ we have

$$L_X(L_Y(f))(p) = \sum_{i=1}^m X_i(p)\partial_i(p)(L_Y(f))$$

where

$$\begin{aligned}\partial_i(p)(L_Y(f)) &= \sum_{j=1}^m \partial_i(p)(Y_j \cdot \partial_j f) \\ &= \sum_{j=1}^m \partial_i(p)Y_j \cdot \partial_j(p)(f) + \sum_{j=1}^m Y_j(p)\frac{\partial^2(f \circ \varphi^{-1})}{\partial x_j \partial x_i}(\varphi(p))\end{aligned}$$

We therefore have

$$L_X(L_Y(f))(p) = \sum_{i,j} X_i(p)\partial_i(p)(Y_j)\partial_j(p)(f) + \sum_{i,j} X_i(p)Y_j(p)\frac{\partial^2(f \circ \varphi^{-1})}{\partial x_j \partial x_i}(\varphi(p))$$

and

$$L_Y(L_X(f))(p) = \sum_{i,j} Y_i(p)\partial_i(p)(X_j)\partial_j(p)(f) + \underbrace{\sum_{i,j} Y_i(p)X_j(p)\frac{\partial^2(f \circ \varphi^{-1})}{\partial x_j \partial x_i}(\varphi(p))}_{\sum_{i,j} Y_j(p)X_i(p)\frac{\partial^2(f\circ\varphi^{-1})}{\partial x_i \partial x_j}(\varphi(p))}$$

By Schwarz's theorem, we thereby conclude

$$(L_{[X,Y]}f)(p) = \sum_{j=1}^m \left(\sum_{i=1}^m X_i(p)\partial_i(p)(Y_j) - Y_i(p)\partial_i(p)(X_j)\right)\partial_j(p)(f)$$

That is

$$[X,Y](p) = \sum_{j=1}^{m} Z_j(p)\partial_j(p)$$

where

$$Z_j(p) = \sum_{i=1}^{m} (X_i(p)\partial_i(p)(Y_j) - Y_i(p)\partial_i(p)(X_j))$$

Note that the value of $[X,Y]$ at $p \in M$ does not only depend on the values of X and Y at p, which would make the bracket a linear algebra story, but on the values of X and Y on a neighbourhood of p.

Also observe that the bracket $[X,Y]$ is bilinear, i.e. for all smooth vector fields X_1, X_2, Y on M and scalars $\lambda_1, \lambda_2 \in \mathbb{R}$ we have

$$[\lambda_1 X_1 + \lambda_2 X_2, Y] = \lambda_1[X_1, Y] + \lambda_2[X_2, Y]$$

and similarly for the Y-slot. It seems like this turns $[-,-]$ into a "product". However, it is not associative. Instead it satisfies Jacobi's identity which is the content of the following Proposition.

Proposition 1.19. Let M be a smooth vector field and let X, Y and Z be smooth vector fields on M. Then

$$[X,[Y,Z]] + [Y,[X,Z]] + [Z,[Y,X]] = 0$$

A way to remember this identity is to note that in the second and third term, the entries X, Y and Z are cyclically permuted. It holds true more generally: Given a vector space W and endomorphisms $A, B, C \in \operatorname{End} W$ it is an easy computation.

Another interpretation of the identity is the following: Given a smooth vector field X on M, define

$$\operatorname{ad}(X) : \Gamma(\mathrm{T}M) \to \Gamma(\mathrm{T}M),\ Y \mapsto [X,Y].$$

Then $\operatorname{ad}(X) \in \operatorname{End}(\Gamma(\mathrm{T}M))$ and the Jacobi identity amounts to $\operatorname{ad}(X)$ preserving brackets: For all $X_1, X_2 \in \Gamma(\mathrm{T}M)$ we have

$$\operatorname{ad}([X_1, X_2]) = [\operatorname{ad} X_1, \operatorname{ad} X_2].$$

There is also a geometric interpretation of the Jacobi identity in terms of flows of the occurring vector fields which we exhibit later.

1.3.2. *Vector Fields on* $\mathbb{R}^n$. Let $\Omega \subseteq \mathbb{R}^n$ be an open set. A smooth vector field on Ω is a smooth map $X : \Omega \to \mathbb{R}^n$. Note however, that the mental picture of a vector field in which the vectors are viewed as being attached to the point. We recall the following existence and uniqueness theorem for integral curves of vector fields in $\mathbb{R}^n$ or rather solutions of ordinary differential equations, see e.g. [Kön13, 4.2 II].

Theorem 1.20. Let $\Omega \subseteq \mathbb{R}^n$ be open and let $Y : \Omega \to \mathbb{R}^n$ be a smooth vector field.

(i) For every $x_0 \in \Omega$ there is an open interval $I_{x_0} \subseteq \mathbb{R}$ containing $0 \in \mathbb{R}$ and an open set $V_{x_0} \subseteq \Omega$ containing $x_0 \in \Omega$ such that for every $x \in V_{x_0}$ there exists a smooth curve $c_x : I_{x_0} \to \Omega$ such that

$$\begin{cases} c_x(0) = x \\ Y(c_x(t)) = c_x'(t) \quad \forall t \in I_{x_0} \end{cases}.$$

(ii) For every $x \in V_{x_0}$, any smooth curve $\gamma : I \to \Omega$ satisfying

$$\begin{cases} \gamma_x(0) = x \\ Y(\gamma_x(t)) = \gamma_x'(t) \quad \forall t \in I_{x_0} \end{cases}$$

coincides with c_x on some neighbourhood of $0 \in \mathbb{R}$.

(iii) The map $V_{x_0} \times I_{x_0} \to \Omega$, $(x,t) \mapsto c_x(t)$ is smooth.

Recall that the key to the proof of Theorem 1.20 consists in transforming an ordinary differential equation into an integral equation, identifying contractivity and applying Banach's fixed point theorem. An analogous statement holds in the case of manifolds.

Corollary 1.21. Let M be a smooth manifold and let $X : M \to \mathrm{T}M$ be a smooth vector field.

(i) For every $x_0 \in M$ there is an open interval $I_{x_0} \subseteq \mathbb{R}$ containing $0 \in \mathbb{R}$ and an open set $V_{x_0} \subseteq M$ containing $x_0 \in M$ such that for every $x \in V_{x_0}$ there exists a smooth curve $c_x : I_{x_0} \to M$ such that

$$\begin{cases} c_x(0) = x \\ X(c_x(t)) = c_x'(t) \quad \forall t \in I_{x_0} \end{cases}.$$

(ii) For every $x \in V_{x_0}$, any smooth curve $\gamma : I \to M$ satisfying

$$\begin{cases} \gamma_x(0) = x \\ X(\gamma_x(t)) = \gamma_x'(t) \quad \forall t \in I_{x_0} \end{cases}$$

coincides with c_x on some neighbourhood of $0 \in \mathbb{R}$.

(iii) The map $V_{x_0} \times I_{x_0} \to M$, $(x, t) \mapsto c_x(t)$ is smooth.

In Corollary 1.21, note that $c_x'(t) := D_t(c_x)(1)$. Its proof consists of expressing everything in local coordinates, applying Theorem 1.20 and transforming back. Later on, we will globalize this local existence and uniqueness statement in the case of compact manifolds.

Retain the notation of Corollary 1.21. For every $t \in I_{x_0}$ we may define the map $\theta_t : V_{x_0} \to M$, $x \mapsto c_x(t)$. The following interesting local group property of the maps θ_t is due to the uniqueness part of Corollary 1.21. Taking the necessary precautions, its proof is mostly formal.

Corollary 1.22. Retain the above notation. Let t_1, t_2 and $t_1 + t_2$ be in I_{x_0}. Further, let $W \subseteq V_{x_0}$ be such that $\theta_{t_1}(W) \subseteq V_{x_0}$. Then $\theta_{t_2} \circ \theta_{t_1}$ and $\theta_{t_1+t_2}$ are defined and agree on W.

Proof. Without loss of generality, we assume $t_1, t_2 \geq 0$. Given $x \in W$ we consider the curve $\gamma(s) := c_x(t_1 + s)$ for $s \in [0, t_2]$. Setting $T : \mathbb{R} \to \mathbb{R}$ to denote the translation $s \mapsto t_1 + s$ we can rewrite $\gamma(s) = c_x \circ T(s)$. We now compute

$$\begin{aligned} \gamma'(s) = (D_s\gamma)(1) = D_s(c_x \circ T)(1) &= D_{T(s)}c_x \circ D_sT(1) \\ &= D_{T(s)}c_x(1) = X(c_x(T(s))) = X(\gamma(s)). \end{aligned}$$

Now observe that $\gamma(0) = c_x(t_1)$. Hence, by uniqueness, $\gamma(s) = c_{c_x(t_1)}(s)$. In particular, for $s = t_2$ we obtain for all $x \in W$:

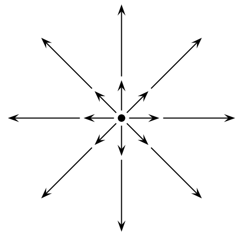
$$\theta_{t_1+t_2}(x) = c_x(t_1 + t_2) = \gamma(t_2) = c_{c_x}(t_2) = \theta_{t_2}(c_x(t_1)) = \theta_{t_2}(\theta_{t_1}(x)).$$

Hence the assertion. □

Later on, we will revisit this result in the case of Lie groups where it takes a more global form.

Example 1.23. Let $M = \mathbb{R}^n$ and consider the radial vector field $X(x) := \sum_{i=1}^n x_i \partial_i(x)$. To determine its integral curves c, note that the condition $X(c(t)) = c'(t)$ translates to $c_i'(t) = c_i(t)$ where $c(t) = (c_1(t), \ldots, c_n(t))$. Consequently $c(t) = k_i e^t$. Taking into account an initial condition $c_x(0) = x$ we get $c_x(t) = e^t x$ and therefore $\theta_t(x) = e^t x$.

One checks that indeed $\theta_{t_1+t_2} = \theta_{t_2} \circ \theta_{t_1}$ in this case.



Example 1.24. To illustrate that the precautions taken in Corollary 1.22 are indeed necessary, consider the case where $M = \mathbb{R}$ and $X(x) := x^2 \partial_x(x)$. The associated initial value problem is $c_x'(t) = c_x^2(t)$ with $c_x(0) = x$. Taking the physicist's approach to solving this, we obtain

$$\frac{-c'}{c^2} = \left(\frac{1}{c}\right)' = -1 \quad \Rightarrow \quad \frac{1}{c} = -t + K$$

and hence $c(t) = 1/(-t+K)$. Therefore, $c_x(t) = 1/(-t+1/x)$. Assuming $x > 0$ the maximum interval of definition of c_x is $(-\infty, 1/x)$. In particular, there is no uniform interval $(-\varepsilon, \varepsilon)$ of definition that works for all initial values.

A possible condition to ensure that the maps θ_t are defined on the whole manifold for all times is compact support of the underlying vector field as in the following proposition.

Proposition 1.25. Let M be a smooth manifold and let $X \in \Gamma(\mathrm{T}M)$ be a smooth vector field with compact support. Then the map, $\mathbb{R} \to \mathrm{Diff}(M)$, $t \mapsto \theta_t$ is a homomorphism.

Homomorphisms as in Proposition 1.25 are termed *one-parameter subgroups of diffeomorphisms* for obvious reasons.

Proof. Let $K = \mathrm{supp}(X) = \overline{\{x \in M \mid X(x) \neq 0\}}$. Using Corollary 1.21 we may cover K with open sets $V_1, \dots, V_l$ such that there are intervals $I_1, \dots, I_l$ to the extent that for every $x \in V_i$ there is a solution $c_x : I_i \to M$ of the initial value problem $X(c_x(t)) = c_x'(t)$, $c_x(0) = x$. Choose $\varepsilon > 0$ such that $(-\varepsilon, \varepsilon) \subseteq \bigcap_{i=1}^l I_i$ and set $\Omega := V_1 \cup \dots \cup V_l$. Then for every $t \in (-\varepsilon, \varepsilon)$ the map $\theta_t : \Omega \to M$ is defined. Outside K, nothing happens: If $x \in M \backslash K$ then $c_x(t) = x$ is a solution to $X(c_x(t)) = c_x'(t)$, $c_x(0) = x$ for all $t \in \mathbb{R}$. In this way, we obtain a map $\theta_t : M \to M$ defined on the whole of M for all $t \in (-\varepsilon, \varepsilon)$. We now use Corollary 1.22 to extend θ_t to all times: First of all, $\mathrm{Id} = \theta_{t-t} = \theta_{-t} \circ \theta_t$ and hence $\theta_t \in \mathrm{Diff}(M)$. Secondly, given $t \in \mathbb{R}$, let $t = k \cdot (\varepsilon/2) + r$ where $k \in \mathbb{Z}$ and $r \in (-\varepsilon, \varepsilon)$; then set $\theta_t := (\theta_{\varepsilon/2})^k \circ \theta_r \in \mathrm{Diff}(M)$. It remains to verify that $\theta_{t_1+t_2} = \theta_2 \circ \theta_1$ which can be done using additivity for small values of t_1, t_2 and $t_1 + t_2$. □

The flows associated to vector fields are very interesting. Although they distort the area started with they sometimes act volume-preservingly. In this case the following may hold: Given a subset $A \subseteq M$, consider the set $\{0 \leq t \leq T \mid \theta_t(x) \in A\} \subseteq \mathbb{R}$. Taking its volume, normalizing by T and letting T tend to infinity, Birkhoff's ergodic theorem states that

$$\lim_{T \to \infty} \frac{1}{T} L(\{0 \leq t \leq T \mid \theta_t(x) \in A\}) = \mathrm{vol}(A)$$

Chaotic behaviour and return to the starting point is typical in particular in the case of compact manifolds. For instance, given an irrationally oriented vector field on the two-torus, the flow always returns to any set with positive measure.

Corollary 1.26. Let M be a smooth compact manifold and let $X \in \Gamma(\mathrm{T}M)$ be a smooth vector field on M. Then there is a one-parameter group of diffeomorphisms $\mathbb{R} \to \mathrm{Diff}(M)$, $t \mapsto \theta_t$ such that for every $x \in M$, the map $\mathbb{R} \to M$, $t \mapsto \theta_t(x)$ is the integral curve of X passing through x at time $t = 0$.

We end the discussion of vector fields by introducing the push-forward of a vector field by a diffeomorphism, relating it the various other operations we have introduced and giving a geometric interpretation of the Lie bracket.

Whereas differential forms can be pulled back their dual objects, vector fields, are pushed forward. However, not every smooth map can be used to do this. For instance, a smooth map might collapse a curve to a point, preventing a natural choice for the push-forward vector at said point. We therefore consider the following situation.

Definition 1.27. Let M and N be smooth manifolds and let $X \in \Gamma(\mathrm{T}M)$ be a smooth vector field. Further, let $\phi : M \to N$ be a diffeomorphism. The *push-forward vector field* Y on N is given by $Y(\phi(y)) := (D_x\phi)(X(x))$.

Lemma 1.28. Retain the notation of Definition 1.27. Then Y is a smooth vector field. Furthermore, if L_X and L_Y denote the derivatives associated to X and Y respectively then $L_Y f = L_X(f \circ \phi) \circ \phi^{-1}$ for all $f \in C^\infty(N)$.

Given a smooth map $\phi : M \to N$ of smooth manifolds, recall that the the associated algebra homomorphism $\phi^* : C^\infty(N) \to C^\infty(M)$ is given by $\phi^*(f) = f \circ \phi$. In terms of ϕ^*, Lemma 1.28 states that the following diagram is commutative.

$$\begin{array}{ccc} C^\infty(N) & \xrightarrow{\phi^*} & C^\infty(M) \\ \Big\downarrow{\scriptstyle L_Y} & & \Big\downarrow{\scriptstyle L_X} \\ C^\infty(N) & \xrightarrow[\phi^*]{} & C^\infty(M) \end{array}$$

With this in mind, the proof of Lemma 1.28 amounts to a computation.

Proof. (Lemma 1.28). Compute for all $f \in C^\infty(N)$:

$$\begin{aligned}(L_Y f)(y) &= Y(y)f = (D_y f)(Y(y)) = (D_y f)(D_x\phi(X(x))) = (D_{\phi(x)} f)(D_x\phi(X(x))) \\ &= D_x(f \circ \phi)(X(x)) = L_X(f \circ \phi)(x) = L_X(f \circ \phi)(\phi^{-1}(y))\end{aligned}$$

□

The relation between push-forward vector fields and the bracket operation is the following.

Lemma 1.29. Let M and N be smooth manifolds and let $\phi : M \to N$ be a diffeomorphism. Further, let X and Z be smooth vector fields on M. Then

$$[\phi_*(X), \phi_*(Z)] = \phi_*([X, Z]).$$

Proof. Here, it is beneficial to think in terms of the associated derivations: We have

$$L_{\phi_* X} = (\phi^*)^{-1} L_X \phi^* \quad \text{and} \quad L_{\phi_*(Z)} = (\phi^*)^{-1} L_Z \phi^*.$$

Computing brackets yields

$$[L_{\phi_* X}, L_{\phi_* Z}] = (\phi^*)^{-1} L_X L_Z \phi^* - (\phi^*)^{-1} L_Z L_X \phi^* = (\phi^*)^{-1} [L_X, L_Z] \phi^*.$$

Hence the assertion. □

Next up is the relation between push-forward vector fields and flows which relies on a uniqueness rather than computation argument.

Lemma 1.30. Let M and N be smooth manifolds and let $\phi : M \to N$ be a diffeomorphism. Further, let X be a smooth vector field on M. Set $Y := \phi_* X$ and let ψ_t denote the flow of Y. Then $\psi_t = \phi\theta_t\phi^{-1}$ for all $t \in \mathbb{R}$ where θ_t is the flow of X.

Proof. Set $\widetilde{\psi}_t(y) := \phi\theta_t\phi^{-1}(y)$. Then

$$\left.\frac{d}{dt}\right|_{t=0} \widetilde{\psi}_t(y) = D_{\phi^{-1}(y)}\phi(X(\phi^{-1}(y))) = D_x\phi(X(x)) = Y(y)$$

which implies the assertion by uniqueness. □

We end this chapter with the following geometric interpretation of the bracket.

Proposition 1.31. Let M be a smooth manifold and let X and Y be smooth vector fields on M. Assume that the local group θ_t of Y is defined. Then

$$\left.\frac{d}{dt}\right|_{t=0} ((\theta_t)_* X) = [X, Y].$$

Note that the expression $((\theta_t)_* X)(x)$ implicit in Proposition 1.31 is a tangent vector at $x \in M$, namely the derivative of X with respect to Y at $x \in M$.

Proof. As an exercise, recall that if $f : (-\varepsilon, \varepsilon) \times M \to \mathbb{R}$ is smooth with $f(0, p) = 0$ for all $p \in M$ then there is a smooth map $g : (-\varepsilon, \varepsilon) \times M \to \mathbb{R}$ with $f(t, p) = tg(t, p)$.

Given $f \in C^\infty(M)$, we apply this fact to $f(t, p) := f(\theta_t(p)) - f(p)$ so that $g(0, p) = Y(p)(f)$. We have

$$\begin{aligned} ((\theta_t)_* X)(p)(f) &= L_X(f \circ \theta_t) \circ \theta_t^{-1}(p) \\ &= X(\theta_t^{-1}(p))(f \circ \theta_t) \\ &= X(\theta_t^{-1}(p))(f + tg(t, -)). \end{aligned}$$

We therefore get

$$\begin{aligned} \frac{(\theta_t)_* X(p)(f) - X(p)(f)}{t} &= \frac{X(\theta_t^{-1}(p))(f) - X(p)(f)}{t} + X(\theta_t^{-1}(p))(g(t, -)) \\ &= \frac{X(\theta_{-t}(p))(f) - X(p)(f)}{t} + X(\theta_{-t}(p))(g(t, -)) \end{aligned}$$

If t tends to zero, we obtain

$$\frac{X(\theta_{-t}(p))(f) - X(p)(f)}{t} \to -L_Y L_X(f)(p)$$

and

$$X(\theta_{-t}(p))(g(t, -)) = L_X(g(t, -))(\theta_{-t}(p)) \to L_X L_Y(f)(p)$$

which implies the assertion. □

1.4. **Lie Groups: A very short introduction.** Lie groups are particularly interesting manifolds and are central to both building large classes of examples of manifolds and actions on manifolds. They are also crucial to seemingly unrelated mathematics such as Fermat's last theorem.

Definition 1.32. A *Lie group* is a smooth manifold G endowed with a group structure such that the product map $G \times G \to G$, $(x, y) \mapsto xy$ and the inverse map $G \to G$, $x \mapsto x^{-1}$ are smooth.

We have already seen many example of Lie groups.

Example 1.33.

(i) The manifold $\mathbb{R}^n$ is a Lie group with respect to addition.
(ii) The manifold $\mathrm{GL}(n, \mathbb{R})$ is a Lie group with respect to matrix multiplication.

(iii) Given a quadratic form q on $\mathbb{R}^n$, the manifold $\mathrm{O}(q)$ is a Lie group with respect to matrix multiplication.
(iv) The symplectic groups of example 1.3 are Lie groups as well.
(v) It is a highly non-trivial result of E. Cartan that every *closed* subgroup $G \leq \mathrm{GL}(n, \mathbb{R})$ is a regular submanifold and hence a Lie group. Note the sharp contrast to the fact that both closed subsets and non-closed subgroups of $\mathrm{GL}(n, \mathbb{R})$ can behave very badly: Consider for instance $\mathrm{GL}(n, \mathbb{Q})$ which can only be turned into a Lie group by making the topology discrete.

We now illustrate that all the notions we have developed work together nicely in the case of Lie groups and thereby underline the importance of the latter. First of all, note that if G is a Lie group and $g \in G$, the left multiplication $L_g : G \to G$, $x \mapsto gx$ is a preferred diffeomorphism of G sending the identity element $e \in G$ to $g \in G$. Indeed, it is smooth by the axioms of a Lie group and admits the smooth inverse $L_{g^{-1}}$. Furthermore, we have the following.

Proposition 1.34. Let G be a Lie group. Then

(i) G is orientable, and
(ii) G is parallelizable.

Proof. To show that G is orientable, we construct a nowhere vanishing top form on G: Let $\omega_0 \in \Lambda^n((\mathrm{T}_e G)^*)$ be a non-zero, alternating n-form where $n = \dim G$ - recall that if V is an n-dimensional vector space then $\dim \Lambda^k(V^*) = \binom{n}{k}$, in particular $\dim \Lambda^n(V^*) = 1$. We now propagate this form to the whole of G using left multiplication: For $g \in G$, set

$$\omega_g(v_1, \ldots, v_n) = \omega_e(D_g L_{g^{-1}} v_1, \ldots, D_g L_{g^{-1}} v_n).$$

One checks that $\omega \in \Omega^n(G)$ so defined is smooth. By construction it is nowhere vanishing, i.e. volume form. Hence G is orientable. Let us endow G with the orientation for which a basis $e_1, \ldots, e_n$ is positively oriented if and only if $\omega_g(e_1, \ldots, e_n) > 0$.

Parallelizabiliy of G is proven as in the case of the torus, see 1.11. □

Corollary 1.35. Let G be a Lie group and let ω denote the volume form on G constructed in the proof of Proposition 1.34. Then the linear map $I : C_{00}(G) \to \mathbb{R}$, defined on the space of continuous, compactly supported functions on G is by $I(f) := \int_G f\ \omega$ yields a left-invariant positive Radon measure on G via Riesz representation.

Let μ denote the measure obtained on a Lie group via Corollary 1.35. It is called *Haar measure* and satisfies $\mu(gE) = \mu(E)$ for every Borel set $E \subseteq G$ and every element $g \in G$. Haar measures exist more generally for locally compact Hausdorff groups. However, in the general case there is a no smooth structure to work with and hence the proof is based on different ideas.

Proof. (Corollary 1.35). We prove invariance of the functional I from which invariance of the associated measure follows. Let $f \in C_{00}(G)$ and $h \in G$. Observe that $(f \circ L_h)\omega = (L_h^*)(f \cdot (L_{h^{-1}}^* \omega)$ and hence

$$I(f \circ L_h) = \int_G (f \circ L_h)\omega = \int_G L_h^* \eta = \int_G \eta$$

where $\eta := f \cdot (L_{h^{-1}}^* \omega$. Now compute

$$\begin{aligned}((L_{h^{-1}})^*\omega)_g(v_1,\ldots,v_n) &= \omega_{L_{h^{-1}}(g)}(D_g L_{h^{-1}}(v_1),\ldots,D_g L_{h^{-1}}(v_n))\\ &= \omega_{h^{-1}g}(D_g L_{h^{-1}}(v_1),\ldots,D_g L_{h^{-1}}(v_n))\\ &= \omega_0(D_{h^{-1}g}L_{g^{-1}h}D_g L_{h^{-1}}(v_1),\ldots,D_{h^{-1}g}L_{g^{-1}h}D_g L_{h^{-1}}(v_n))\\ &= \omega_0(D_g L_{g^{-1}}v_1,\ldots,D_g L_{g^{-1}}v_n)\\ &= \omega_g(v_1,\ldots,v_n).\end{aligned}$$

whence $\eta = f\omega$ and therefore $I(f \circ L_h) = I(f)$. □

Observe that if G is a compact Lie group then the functional I of Corollary 1.35 is defined for all $f \in C(G)$.

Corollary 1.36. Every compact Lie subgroup of $\mathrm{GL}(n,\mathbb{R})$ is conjugate into $\mathrm{O}(n)$.

Proof. Let $K \leq \mathrm{GL}(n,\mathbb{R})$ be a compact Lie subgroup and let μ denote its Haar measure. Choose any scalar product $\langle -,-\rangle$ on $\mathbb{R}^n$ and define $B(v,w) := \int_G \langle gv, gw\rangle \, d\mu(g)$ for all $v,w \in \mathbb{R}^n$. Then it is an easy verification that B is a K-invariant scalar product. Now remember from linear algebra that if $(-,-)$ denotes the standard scalar product then there exists $A \in \mathrm{GL}(n,\mathbb{R})$ such that $B(u,v) = (Au,Av)$ for all $u,v \in \mathbb{R}^n$. Since B is K-invariant we obtain $(Aku,Akv) = (Au,Av)$. Setting $u' := Au$ and $v' := Av$ we conclude $(AkA^{-1}u', AkA^{-1}v') = (u',v')$ and hence $AKA^{-1} \subseteq \mathrm{O}(n)$. □

We now apply the theory that we have developed for manifolds to the special case of Lie groups in which things behave particularly nice.

Definition 1.37. A *Lie algebra* is a vector space $\mathfrak{g}$ endowed with a product $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, denoted $(A,B) \mapsto [A,B]$ which is

(i) bilinear,
(ii) satisfies $[A,B] = -[B,A]$ for all $A,B \in \mathfrak{g}$, and
(iii) satisfies $[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0$ for all $A,B,C \in \mathfrak{g}$.

Example 1.38. Let M be a smooth manifold. Then $\mathfrak{g} = \Gamma(\mathrm{T}M)$, endowed with the bracket of vector fields, is a Lie algebra. However, it is infinite-dimensional unless M is a point.

Definition 1.39. Let G be a Lie group. A smooth vector field $X \in \Gamma(\mathrm{T}G)$ is *left-invariant* if $(L_g)_*X = X$ for all $g \in G$.

Let $\Gamma^{\mathrm{inv}}(\mathrm{T}G)$ be the subspace of $\Gamma(\mathrm{T}G)$ consisting of left-invariant vector fields.

Proposition 1.40. Let G be a Lie group. Then $\Gamma^{\mathrm{inv}}(\mathrm{T}G)$ is a Lie algebra and the map $\Gamma^{\mathrm{inv}}(\mathrm{T}G) \to \mathrm{T}_eG$ given by $X \mapsto X(e)$ is a vector space isomorphism.

In particular, the Lie algebra $\Gamma^{\mathrm{inv}}(\mathrm{T}G)$ is finite-dimensional. In some way, it encodes the group structure of G. However, it is unclear yet in what way it does so.

Proof. In Lemma 1.28 we have shown that $(L_g)_*[X,Y] = [(L_g)_*X, (L_g)_*Y]$ for all $X,Y \in \Gamma(\mathrm{T}G)$ and $g \in G$. Hence, if X and Y are invariant smooth vector fields on G then so is $[X,Y]$.

Injectivity of the evaluation map is due to the fact that left-invariance of X yields $X(g) = (D_eL_g)(X(e))$. Surjectivity follows from defining $Y \in \Gamma^{\mathrm{inv}}(\mathrm{T}G)$ by setting $Y(g) := (D_eL_g)(v)$ for a given $v \in \mathrm{T}_eG$. □

The isomorphism of Proposition 1.40 can be used to turn T_eG into a Lie algebra as follows: Given $u,v \in \mathrm{T}_eG$, define $[u,v] \in \mathrm{T}_eG$ to be the evaluation of left-invariant vector field $[X_u,X_v]$ at e where X_u and X_v are the left-invariant vector fields corresponding to u and v respectively. Note that in order to know $[u, v]$ it does not suffice to know u and v, it requires knowledge of the vector fields X_u and X_v on a small neighbourhood of the identity in G.

One checks that the bracket on the Lie algebra of $\mathbb{R}^n$ is trivial. Generally, it measures the degree of non-commutativity and is best used for highly non-abelian groups, such as semisimple ones.

Definition 1.41. Let G be a Lie group. The *Lie algebra* of G is the vector spcae $\mathfrak{g} := \mathrm{T}_e G$ endowed with the product $[x, y] := [X, Y](e)$ where X and Y are the left-invariant vector fields associated to x and y in $\mathrm{T}_e G$.

In the context of general manifolds we have studied flows of vector fields. There is a lot to say about these in the context of Lie groups.

Proposition 1.42. Let G be a Lie group and let $X \in \Gamma^{\mathrm{inv}}(\mathrm{T}G)$. Then the local group θ_t is defined for all $t \in \mathbb{R}$. In addition,

(i) $\theta_t(g) = g\theta_t(e)$ for all $t \in \mathbb{R}$ and $g \in G$, and
(ii) the map $\mathbb{R} \to G$, $t \mapsto \theta_t(e)$ is a smooth homomorphism whose tangent vector at $e \in G$ is $X(e)$.

Proof. The main part of Proposition 1.42 is to show the existence of integral curves for all times. The remaining statements are then easy consequences. For every $g \in G$, let $I_g \subseteq \mathbb{R}$ be the largest open interval of definition of c_g which contains $0 \in \mathbb{R}$. We are going to show that I_g is independent of g which suffices as we shall see. For once, we claim that $c_g(t) = L_g(c_e(t))$ which can be verified using uniqueness: Let $\gamma(t) := L_g(c_e(t))$ for $t \in I_e$. Then

$$\gamma'(t) = (D_{c_e(t)} L_g)(c_e'(t)) = (D_{c_e(t)} L_g)(X(c_e(t))) = X(gc_e(t)) = X(\gamma(t)).$$

Furthermore, $\gamma(0) = gc_e(0) = ge = g$. hence $\gamma(t) = c_g(t)$ by uniqueness. This proves the claim. In particular, we conclude that $I_g = I_e =: I$ for all $g \in G$. Let $\varepsilon > 0$ be such that $(-\varepsilon, \varepsilon) \subseteq I$ and choose $t \in I$. For small enough s we have $c_g(t+s) = c_{c_g(t)}(s)$ by uniqueness. Now, the right hand side is defined at least for all $s \in (-\varepsilon, \varepsilon)$ which implies $t + (-\varepsilon, \varepsilon) \subseteq I_g = I_e = I$. Hence $I = \mathbb{R}$.

The claim can be rewritten as $\theta_t(g) = g\theta_t(e)$. Setting $g = \theta_s(e)$ we obtain

$$\theta_t \circ \theta_s(e) = \theta_s(e)\theta_t(e).$$

On the other hand, we know that $\theta_t \circ \theta_s = \theta_{t+s}$ and hence $\theta_{t+s}(e) = \theta_s(e) \cdot \theta_t(e)$. In other words, the map $\mathbb{R} \to G$, $t \mapsto \theta_t(e)$ is a homomorphism. □

The proof of Proposition 1.42 leads us to the following, fundamental object.

Definition 1.43. Let G be a Lie group and let $\mathfrak{g}$ be its Lie algebra. The *exponential map* $\exp_G : \mathfrak{g} \to G$ is given by $x \mapsto \theta_1(e)$ where θ_t denotes the one-parameter group of diffeomorphism associated to the left-invariant vector field X with $X(e) = x$.

We observe that as a consequence of Definition 1.43, we have $\theta_t(e) = \exp(tx)$ which amounts to saying the vector fields can be scaled: Indeed, the map $\mathbb{R} \to \mathrm{Diff}(G)$, $s \mapsto \theta_{st}$ is the one-parameter group associated to tX; its value at $s = 1$ being $\exp(tx)$ coincides with $\theta_t(e)$.

We shall see shortly that the exponential map of a Lie group G with Lie algebra $\mathfrak{g}$ is a local diffeomorphism at $0 \in \mathfrak{g}$. That is, Lie groups come equipped with canonical charts that are very well adapted to the group structure. For $(\mathbb{R}^n, +)$ the exponential map merely associates the position vector to a tangent vector. Its name, however, stems from the fact that in the case of $\mathrm{GL}(n, \mathbb{R})$, the exponential map is the usual matrix exponential.

Example 1.44. As an example, we determine the exponential map in the case of $\mathrm{GL}(n,\mathbb{R})$. The manifolds structure of $\mathrm{GL}(n,\mathbb{R})$ stems from the fact that it is an open subset of $M_{n,n}(\mathbb{R})$. We may hence identify $\mathrm{T}_{\mathrm{Id}}\,\mathrm{GL}(n,\mathbb{R})$ with $\mathrm{M}_{n,n}(\mathbb{R})$. Let $x\in M_{n,n}(\mathbb{R})$. For appropriate $\varepsilon>0$, the curve $t\mapsto \mathrm{Id}+tx$ through $\mathrm{Id}\in M_{n,n}(\mathbb{R})$ is contained in $\mathrm{GL}(n,\mathbb{R})$ for all $t\in(-\varepsilon,\varepsilon)$. Let X be the left-invariant vector field on $\mathrm{GL}(n,\mathbb{R})$ associated to $x\in M_{n,n}(\mathbb{R})$. Then

$$X(g)=(D_eL_g)(x)=\left.\frac{d}{dt}\right|_{t=0} g(\mathrm{Id}+tx)=gx.$$

From this we deduce that $(L_Xf)(g)=X(g)(f)=(D_gf)(gx)$ for all $f\in C^\infty(G)$ and $g\in\mathrm{GL}(n,\mathbb{R})$. A computation now implies that for all $x,y\in M_{n,n}(\mathbb{R})=\mathrm{T}_{\mathrm{Id}}\,\mathrm{GL}(n,\mathbb{R})$ we have $[x,y]=xy-yx$. That is, in this case, the Lie bracket coincides with the usual commutator bracket operation.

Now, the differential equation given by an invariant vector field X is $X(c(t))=c'(t)$, that is $c(t)x=c'(t)$. Its solution with initial condition $c(0)=\mathrm{Id}$ is given by $c(t)=\mathrm{Exp}(tx)=\sum_{k=0}^{\infty}(t^kx^k)/k!$. Thus $\exp_{\mathrm{GL}(n,\mathbb{R})}(x)=\mathrm{Exp}(x)$ is the usual matrix exponential, hence the name of the general Lie group exponential.

Proposition 1.45. Let G be a Lie group with Lie algebra $\mathfrak{g}$. The exponential map $\exp_G:\mathfrak{g}\to G$ is smooth and $D_0\exp_G:\mathfrak{g}\to\mathfrak{g}$ is the identity map.

Proof. Identifying $\mathrm{T}_0\,\mathfrak{g}$ with $\mathfrak{g}$ we have $D_0\exp_G(x)=\frac{d}{dt}\big|_{t=0}\exp_G(tx)=x$. □

The importance of Proposition 1.45 is due to the fact that it implies that the exponential map is a diffeomorphism on a neighbourhood of $0\in\mathfrak{g}$. Hence its inverse can be employed as a chart.

Lie subgroups. It is also essential that the exponential map of a Lie group G determines the exponential maps of all its Lie subgroups H. The difficulty of this statement lies in the definition of *Lie subgroup.* Whereas this chapter does not aim at being a comprehensive treatment of the foundations of Lie theory, we elaborate a bit on this important point: Let G be a Lie group and let $H\le G$ be a subgroup. If H is also a regular submanifold and the map $H\times H\to H$, $(x,y)\mapsto xy^{-1}$ is smooth then H is a Lie group. In this case, one can verify that $\mathfrak{h}:=\mathrm{T}_eH\le\mathrm{T}_eG=\mathfrak{g}$ is a subalgebra. In general, however, we do not get every subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ in this fashion which is unacceptable from a categorical point of view. For instance, consider the torus $\mathbb{T}^2=\mathbb{Z}^2\setminus\mathbb{R}^2$ whose Lie algebra is $\mathrm{Lie}(\mathbb{T}^2)=\mathbb{R}^2$ has trivial bracket. Consequently, all subspaces of $\mathrm{Lie}(\mathbb{T}^2)$ are subalgebras. But the subalgebra generated by $(1,\sqrt{2})^T\in\mathbb{R}^2$ is not the Lie algebra of a regularly embedded subgroup of $\mathbb{T}^2$.

In fact, one has to relax the requirement of the subgroup to be regularly embedded: In general, a *Lie subgroup* of a Lie group G is a pair (H,i) consisting of a Lie group H and an injective immersion $i:H\to G$, allowing for the topologies on H and $i(H)\subseteq G$ to differ as in the above case.

Proposition 1.46. Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let $H\le G$ be a regular submanifold. Then the Lie algebra of H is given by

$$\mathfrak{h}=\mathrm{T}_eH=\{v\in\mathrm{T}_eG=\mathfrak{g}\mid \exp_G(tv)\in H\ \forall t\in\mathbb{R}\}.$$

Example 1.47. Proposition 1.46 allows us to compute the Lie algebras of most our examples, which are subgroups of $\mathrm{GL}(n,\mathbb{R})$. In this case, the general recipe is to write down the defining equation of the group, substitute $\exp(tx)$ for the variable, and take the derivative at $t=0$ to obtain the defining equation of the Lie group.

(i) In the case of $\mathrm{O}(n)$ we compute

$$\begin{aligned}\mathrm{Lie}(\mathrm{O}(n)) &= \{x \in M_{n,n}(\mathbb{R}) \mid \mathrm{Exp}(tx) \in \mathrm{O}(n)\ \forall t \in \mathbb{R}\}\\ &= \{x \in M_{n,n}(\mathbb{R}) \mid \mathrm{Exp}(tx)^T \,\mathrm{Exp}(tx) = \mathrm{Id}\ \forall t \in \mathbb{R}\}\\ &= \{x \in M_{n,n}(\mathbb{R}) \mid x^T + x = 0\}\end{aligned}$$

as $(d/dt)|_{t=0}(\mathrm{Exp}(tx)^T \,\mathrm{Exp}(tx)) = x^T + x$.

(ii) Similar to the above, we determine in the case of $\mathrm{Sp}(2n, \mathbb{R})$:

$$\mathrm{Lie}(\mathrm{Sp}(2n, \mathbb{R})) = \{x \in M_{2n,2n}(\mathbb{R}) \mid x^T J + Jx = 0\}.$$

It is an exercise to think about what happens in cases such as $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ as subgroups of $\mathrm{GL}(n, \mathbb{C})$.

In a sense *linear groups* such as the examples above already cover almost all Lie groups. This is made precise using representations of Lie groups: Every Lie group G comes equipped with a natural action on a finite-dimensional vector space: Recall that for $h \in G$, we have the right-multiplication diffeomorphism $R_h : G \to G$ given by $g \mapsto gh$. Any R_h commutes with any L_g $(g, h \in G)$ which is due to the easily overlooked associativity of the group multiplication. Therefore, given $X \in \Gamma^{\mathrm{inv}}(\mathrm{T}G)$ and $h \in G$, we have $(R_h)_*X \in \Gamma^{\mathrm{inv}}(\mathrm{T}G)$. Accordingly, we may define for $h \in G$ and $x \in \mathfrak{g} = \mathrm{T}_eG$: $\mathrm{Ad}(h)x := ((R_{h^{-1}})_*X)(e)$. In this way, we get a homomorphism $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$, called the *adjoint representation*. The kernel of the adjoint representation is related to the center of G and is abelian, which is not considered a drama because abelian Lie groups are well-understood.

We end this chapter on Lie groups with the two following results on the adjoint representation of a Lie group.

Proposition 1.48. Let G be a Lie group with Lie algebra $\mathfrak{g}$. Then for all $h \in G$ and $x \in \mathfrak{g}$ we have

$$\mathrm{Ad}(h)(x) = \left.\frac{d}{dt}\right|_{t=0} h \exp(tx) h^{-1}.$$

The proof of Proposition 1.48 is left as an exercise. Now, the adjoint representation of a Lie group G is a map from G to $\mathrm{GL}(\mathfrak{g})$. As such it can be differentiated at $t = 0$ to produce a homomorphism of the associated Lie algebras $\mathfrak{g}$ and $\mathrm{End}(\mathfrak{g})$. This produces the following interpretation of the Lie bracket.

Proposition 1.49. Let G be a Lie group with Lie algebra $\mathfrak{g}$. For all $x, y \in \mathfrak{g}$ we have

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}(\exp ty)(x) = [y, x].$$

The proof of Proposition 1.49 is a consequence of Proposition 1.31.

1.5. **Coverings and Fibrations.** In this section we discuss the important covering and fibration mechanisms. In the following definition, we adopt the topological notion of covering maps to our smooth setting.

Definition 1.50. Let $\overline{M}$ and M be smooth manifolds. A map $p : \overline{M} \to M$ is a *covering map* if

(i) p is smooth and surjective, and
(ii) for every $m \in M$ there is an open neighbourhood U of $m \in M$ such that $p^{-1}(U) = \bigcup_{i \in I} U_i$ is a disjoint union of open subsets $U_i \subseteq \overline{M}$ such that $p|_{U_i} : U_i \to U$ is a diffeomorphism.

Given a covering map $p : \overline{M} \to M$ the map $M \to \mathbb{N}_0 \cup \{\infty\}$ which to $m \in M$ associates the cardinality of its preimage under p is locally constant.

Example 1.51. Here are three of the most important examples and non-examples of covering spaces.

(i) Let $\overline{M} := M := S^1$ and consider $p : \overline{M} \to M$, $z \mapsto z^d$ for some $d \in \mathbb{Z}$ where S^1 is considered as $\{z \in \mathbb{C} \mid |z| = 1\}$. Then p is a d-sheeted covering map and given $z \in M$ we have $p^{-1}(z) = \{\exp(2\pi ik/d) \mid 0 \leq k < d\}$.
(ii) The map $f : \mathbb{C} \to \mathbb{C}$, $z \mapsto z^2$ is not a covering: There is no neighbourhood of $0 \in \mathbb{C}$ on which $\operatorname{card}(p^{-1}(z))$ is constant. However, one could remove 0 from both $\overline{M}$ and M to turn f into a covering.
(iii) Set $\overline{M} := \mathbb{R}$ and $M = S^1$. Then the map $t \mapsto \exp(2\pi it)$ is a covering.

One way in which covering maps arise is the following: Let M be a locally compact Hausdorff space on which a group Γ acts by homeomorphisms. That is, for each $\gamma \in \Gamma$ the map $M \to M$, $m \mapsto \gamma m$ is a homeomorphism.

Definition 1.52. Let M be a locally compact Hausdorff space on which a group Γ acts by homeomorphisms. That is, for each $\gamma \in \Gamma$ the map $M \to M$, $m \mapsto \gamma m$ is a homeomorphism.

(i) The action is *properly discontinuous* if for every compact $K \subseteq M$ the set $\{\gamma \in \Gamma \mid \gamma(K) \cap K \neq \emptyset\}$ is finite.
(ii) The action is *free* if for every $x \in M$ and $\gamma \in G \backslash \{e\}$ we have $\gamma(x) \neq x$.

Proposition 1.53. Let M be a locally compact Hausdorff space on which a group Γ acts by homeomorphisms. If the action is properly discontinuous then $\Gamma \backslash M$ is Hausdorff.

In the case of a group Γ acting on a set $\Gamma \backslash M$, we denote by $\Gamma \backslash M$ the set of equivalence classes for the relation $x \sim y \iff \Gamma x = \Gamma y$. We remark that in this situation the map $p : M \to \Gamma \backslash M$ is open: Indeed, if $V \subseteq M$ is any set, then $p^{-1}(p(V)) = \bigcup_{\gamma \in \Gamma} \gamma(V)$. Hence, if V is open, then $p^{-1}(p(V))$ is open as a union of open sets.

Proof. (Proposition 1.53). Since p is open it suffices to show that $R = \{(x, y) \in M \times M \mid x \sim y\} = \{(x, \gamma x) \mid x \in M,\ \gamma \in \Gamma\}$ is a closed subset of $M \times M$. To this end, let $(x_0, y_0) \in \overline{R}$. For every compact neighbourhood of $(x_0, y_0) \in M \times M$ of the form $V \times W$, define $F(V, W) := \{\gamma \in \Gamma \mid \gamma V \cap W \neq \emptyset\}$. Observe that $F(V, W)$ is always finite: Indeed,

$$F(V, W) \subseteq \{\gamma \in \Gamma \mid \gamma(V \cup W) \cap (V \cup W) \neq \emptyset\}$$

and the latter set is finite by the definition of proper discontinuity. Now, fix a particular neighbourhood $V_0 \times W_0$ and consider

$$\mathcal{F} := \{F(V, W) \mid V \times W \ni (x_0, y_0) \text{ compact neighbourhood and } V \times W \subseteq V_0 \times W_0\}$$

Now make the following observations:

(i) $F(V,W) \subseteq F(V_0,W_0)$ for all $F(V,W) \in \mathcal{F}$.
(ii) $F(V,W) \neq \emptyset$ for all $F(V,W) \in \mathcal{F}$: Indeed, since $(x_0,y_0) \in \overline{R}$ we have $(V \times W) \cap R \neq \emptyset$ and a point $(x,\gamma x) \in (V \times W)$ gives rise to $\gamma \in F(V,W)$.
(iii) Given $F(V_i,W_i) \in \mathcal{F}$ $(i \in \{1,\dots,n\})$, we have

$$\bigcap_{i=1}^{n} F(V_i,W_i) \supseteq F\left(\bigcap_{i=1}^{n} V_i, \bigcap_{i=1}^{n} W_i\right) \neq \emptyset.$$

As a consequence, we conclude $\bigcap_{F(V,W)\in\mathcal{F}} F(V,W) \neq \emptyset$. An element $\gamma \in \Gamma$ that is contained in the latter intersection has the property that for any compact neighbourhoods V of $x_0 \in M$ and W of $y_0 \in M$ we have $\gamma V \cap W \neq \emptyset$. Since M is Hausdorff this implies $\gamma(x_0) = y_0$. □

Corollary 1.54. Let Γ be a group acting freely and properly discontinuously on a manifold M by diffeomorphisms. Then there is a unique smooth manifold structure on $\Gamma\backslash M$ such that $p : M \to \Gamma\backslash M$ is a smooth covering.

Proof. (Sketch). Proper discontinuity and freeness of the action imply that for all $x \in M$ there is an open neighbourhood V_x of x such that for all $\gamma \in \Gamma\backslash\{e\}$ we have $\gamma V_x \cap V_x = \emptyset$. This is based on the following argument: Given $\gamma \in \Gamma\backslash\{e\}$, pick disjoint neighbourhoods V of x and W of γx with $V \cap W = \emptyset$ using that M is Hausdorff. Then $V \cap \gamma^{-1}W$ is a neighbourhood of x and $(V\cap\gamma^{-1}W)\cap(\gamma V\cap W) = \emptyset$. Adjust this argument to taking into account the finiteness of our situation.

For a neighbourhood V_x of x as above, we observe that $p|_{V_x} : V_x \to p(V_x)$ is continuous, open, surjective and injective, hence a homeomorphism.

Finally, choose an atlas $\mathcal{A}$ on M consisting of charts (V_x,φ_x) as above. A smooth atlas on $\Gamma\backslash M$ is then given by

$$\mathcal{A}' = \{(pV_x,\phi_x) \mid (V_x,\phi_x) \in \mathcal{A},\ \phi_x = \varphi_x \circ (p|_{V_x})^{-1}\}.$$

The smoothness of the transition maps comes from the assumption that Γ acts on M by diffeomorphisms. □

Example 1.55. We collect several examples of this efficient construction.
(i) Let $\Gamma = \{\pm \mathrm{Id}\}$ and $M = S^n \subseteq \mathbb{R}^{n+1}$. Then $\Gamma\backslash S^n$ is diffeomorphic to real projective n-space $\mathbb{P}^n(\mathbb{R})$.
(ii) Consider $M := H^+ := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$. One verifies that the map

$$\mathrm{SL}(2,\mathbb{R}) \times M \to M,\ \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z\right) \mapsto \frac{az+b}{cz+d}$$

defines an action of $\mathrm{SL}(2,\mathbb{R})$ on H^+. The reader is encouraged to show that $\Gamma := \mathrm{SL}(2,\mathbb{Z})$, consisting of all the matrices in $\mathrm{SL}(2,\mathbb{R})$ with integer entries, acts properly discontinuously on M via the above action (in contrast to having dense orbits on the real line).
(iii) Let G be a Lie group and let $\Gamma \leq G$ be a discrete subgroup. Then the left action $\Gamma \times G \to G$, $(\gamma,g) \mapsto \gamma g$ of Γ on G is free and properly discontinuous. Hence $\Gamma\backslash G$ admits a smooth manifold structure for which $p : G \to \Gamma\backslash G$ is a covering.

For the upcoming discussion of fibrations, we record the following: Given compact subsets K_1 and K_2 of a Lie group G, the set $K_2K_1^{-1} = \{k_2k_1^{-1} \mid k_2 \in K_2,\ k_1 \in K_1\}$ is compact: Indeed, it is the image of the compact set $K_1 \times K_2 \subseteq G \times G$ under the smooth map $G \times G \to G$ given by $(x,y) \mapsto yx^{-1}$.

Fibrations are a common generalization of the concepts of vector bundles and coverings.

Definition 1.56. Let E, B and F be smooth manifolds and let $\pi : E \to B$ be a smooth map. The triple (π, E, B) is a *fiber bundle with base B, total space E and fiber F* if

(i) the map $\pi : E \to B$ is surjective, and
(ii) there is an open cover $\{U_i \mid i \in I\}$ of B and diffeomorphisms

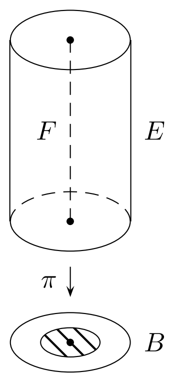
$$h_i : \pi^{-1}(U_i) \to U_i \times F \text{ with } h_i(\pi^{-1}(x)) = \{x\} \times F.$$

Example 1.57.

(i) Coverings yield fiber bundles with discrete fiber.
(ii) Vector bundles are fibrations with fiber a vector space.
(iii) Consider the following interpretation of $S^3 \subseteq \mathbb{R}^4$:

$$E = \{(z_1, z_2) \in \mathbb{C}^2 \mid z_1^2 + z_2^2 = 1\}$$

The group $S^1 = \{\varrho \in \mathbb{C} \mid |\varrho| = 1\}$ acts on E via $\varrho(z_1, z_2) = (\varrho z_1, \varrho z_2)$. This action is free and the quotient identifies with $\mathbb{P}^1(\mathbb{C})$. The latter is incidentally diffeomorphic to S^2. This is a fibration of S^3 over S^2 with fiber S^1, the so called Hopf fibration.



Fibrations are particularly fruitful in the setting of Lie groups acting on manifolds: An action of a Lie group G on a manifold M is *smooth* if the action map $G \times M \to M$ is smooth. In particular, for every $g \in G$ the map $M \to M$, $m \mapsto gm$ is a diffeomorphism: It is smooth as a restriction of the smooth action map and has an inverse arising in the same fashion. The action of G on M is *proper* if for every compact set $K \subseteq M$ the set $\{g \in G \mid gK \cap K \neq \emptyset\}$ has compact closure in G.

Theorem 1.58. Let G be a Lie group, M a manifold and $G \times M \to M$ a smooth action of G on M. If the action is free and proper then there is a unique smooth manifold structure on $G\backslash M$ such that $M \to G\backslash M$ is a smooth fibration.

A good reference for this Theorem is [vdB06], in which Theorem is deduced from the following, more general equivalence relation version which mostly depends on the inverse function theorem.

Theorem 1.59. Let M be a smooth manifold and let $R \subseteq M \times M$ be an equivalence relation on M. If R is a closed submanifold of $M \times M$ and $\mathrm{pr}_1 : R \to M$ is a submersion then the quotient $R\backslash M$ has a unique structure of smooth manifold such that $\pi : M \to R\backslash M$ is a submersion.

In particular, we can apply Theorem 1.58 in the case where $M = G$ is a Lie group and $H \leq G$ is a closed subgroup of G acting on G on the right: $G \times H \to G$, $(g, h) \mapsto gh$.

Lemma 1.60. Let G be a Lie group and let $H \leq G$ be closed. Then the right action of H on G is proper and free.

Proof. Freeness is immediate. As to properness, let $K \subseteq G$ be compact. Then

$$\{h \in H \mid Kh \cap K \neq \emptyset\} = H \cap K^{-1}K.$$

Since H is closed and $K^{-1}K$ is compact, so is the above intersection. □

As a corollary, we obtain the following powerful mechanism of producing manifolds and actions on them.

Corollary 1.61. Let G be a Lie group and let H be a closed subgroup of G. Then there is a unique smooth manifold structure on G/H such that the quotient map $\pi : G \to G/H$ is a smooth fibration with base G/H and fiber H. With this manifold structure, the action $G \times G/H \to G/H$ is smooth.

Corollary 1.61 produces one of the most enormous classes of manifolds with many interesting subclasses depending on properties of G and H.

Example 1.62. To illustrate the usefulness of Corollary 1.61 we consider the following: Equip $\mathbb{R}^n$ with the usual scalar product $\langle -, - \rangle$. For $1 \le p \le n$, let

$$S_{p,n} := \{(x_1, \dots, x_p) \in (\mathbb{R}^n)^p \mid \langle x_i, x_j \rangle = \delta_{ij} \ \forall i, j \in \{1, \dots, p\}\}.$$

Observe that $\mathrm{O}(n)$ acts transitively on $S_{p,n}$ as it acts transitively on orthonormal bases. Let $(e_1, \dots, e_n)$ be the standard orthonormal basis of $\mathbb{R}^n$ and fix the basepoint $p := (e_1, \dots, e_p) \in S_{p,n}$. The stabilizer of p in $\mathrm{O}(n)$ is given by

$$H_p = \left\{ \begin{pmatrix} \mathrm{Id}_p & 0 \\ 0 & B \end{pmatrix} \middle| \, B \in \mathrm{O}(n-p) \right\}$$

which is a closed, regularly embedded submanifold of $\mathrm{O}(n)$. Hence the bijection

$$\mathrm{O}(n)/H_p \cong \mathrm{O}(n)/\mathrm{O}(n-p)$$

can be used to equip $S_{p,n}$ with a smooth manifold structure. The manifolds $S_{p,n}$ are called Stiefel manifolds and play a fundamental role when it comes to characteristic classes and vector bundles. A similar reasoning as above, lets us treat Grassmannian manifolds as quotients of Lie groups and hence equip them with a manifold structure in a very convenient way.

Combining results from above we record the following.

Corollary 1.63. Let M be a manifold and let $b \in M$. Furthermore, let G be a Lie group acting smootly and transitively on M. Let H denote the stabilizer in G of $b \in M$. Then the bijection $G/H \to M$, $gH \mapsto gb$ is a diffeomorphism.

Corollary 1.63 is pleasant compatibility result and in particular states that one cannot find any exotic smooth structures on a manifold by exhibiting it as a homogeneous space of a Lie group as above.

Example 1.64. We now have a look at further examples of this mechanism.

(i) The Lie group $\mathrm{GL}(n, \mathbb{R})$ acts smoothly and transitively on $\mathbb{R}^n \setminus \{0\}$. The stabilizer of $e_1 = (1, 0, \dots, 0)^T \in \mathbb{R}^n$ in $\mathrm{GL}(n, \mathbb{R})$ is given by

$$P_0 = \left\{ \begin{pmatrix} 1 & v \\ 0 & A \end{pmatrix} \middle| \, v \in \mathbb{R}^{n-1}, \ A \in \mathrm{GL}(n-1, \mathbb{R}) \right\}$$

which can be identified as a semi-direct product. Anyway, we conclude that $\mathrm{GL}(n, \mathbb{R})/P_0$ is diffeomorphic to $\mathbb{R}^n \setminus \{0\}$.

(ii) The Lie group $\mathrm{GL}(n, \mathbb{R})$ also acts on the set $\mathbb{P}^n(\mathbb{R})$ of all lines passing through $0 \in \mathbb{R}^{n+1}$. Recall that $\mathbb{P}^n(\mathbb{R})$ can also be defined as the quotient of $\mathbb{R}^{n+1} \setminus \{0\}$ by the multiplication action of $\mathbb{R}^*$, and the quotient of S^n by the antipodal map. Whereas the action of $\mathrm{GL}(n, \mathbb{R})$ is not visisble in the second definition, the first one clearly shows its transitivity. The stabilizer of $[e_1] \in \mathbb{P}^n(\mathbb{R})$ is given by

$$P := \left\{ \begin{pmatrix} a & v \\ 0 & A \end{pmatrix} \middle| \, a \in \mathbb{R} \setminus \{0\}, \ v \in \mathbb{R}^n, \ B \in \mathrm{GL}(n, \mathbb{R}) \right\}.$$

As a consequence, $\mathrm{GL}(n+1, \mathbb{R})/P$ is diffeomorphic to $\mathbb{P}^n(\mathbb{R})$.

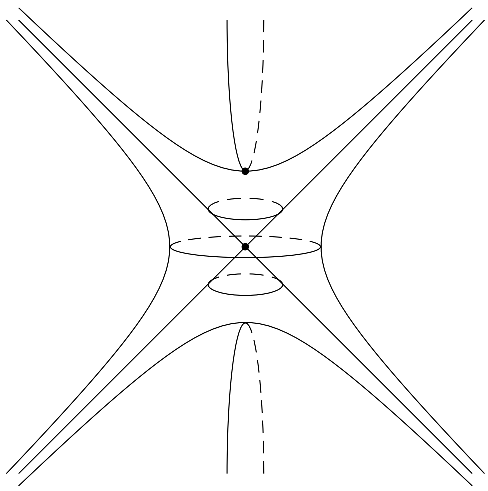
In the constext of (ii), the following exercises are worthwhile: For $g \in \mathrm{GL}(n, \mathbb{R})$, study the qualitative behaviour of the action of $\{g^n \mid n \in \mathbb{Z}\}$ on $\mathbb{P}^n(\mathbb{R})$. Given that linear algebra classifies elements of $\mathrm{GL}(n, \mathbb{R})$ up to conjugacy, three particular examples to look at are

$$\begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix} \ (\lambda_1 \neq \lambda_2), \quad \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \text{ and } \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}.$$

What do the behaviours say towards a classification of homeomorphisms of S^1 up to conjugacy? Finally, one can consider the same question for the case of $\mathrm{GL}(3, \mathbb{R})$ where one discovers new phenomena, semi-hyperbolicity.

Also, does there exist an $\mathrm{SL}(2, \mathbb{R})$-invariant probability measure on $\mathbb{P}^1(\mathbb{R})$? The answer to this question is "no". However, is this a general phenomenon, in the sense that no compact homogeneous space $\mathrm{SL}(2, \mathbb{R})/H$ of $\mathrm{SL}(2, \mathbb{R})$ supports an invariant probability measure? Again, the answer is "no".

(iii) Returning to the examples, consider the following construction which is of fundamental importance for the remainder of the course. Let $q : \mathbb{R}^{n+1} \to \mathbb{R}$ be the quadratic form given by $q(x) = x_1^2 + \cdots + x_n^2 - x_{n+1}^2$. This quadratic form has three kinds of level sets:
  (a) For $a < 0$, the equation $x_1^2 + \cdots + x_n^2 = x_{n+1}^2 + a$ has no solutions with $-\sqrt{-a} < x_{n+1} < \sqrt{-a}$. For $|x_{n+1}| > \sqrt{a}$, there are (rotationally invariant) solution. In fact, we obtain a two-sheeted hyperboloid.
  (b) The level set $q^{-1}(0)$ is a cone with singular point $0 \in \mathbb{R}^{n+1}$.
  (c) For $a > 0$, the level set $q^{-1}(a)$ is a one-sheeted hyperboloid.



Recall that the symmetry group of q is

$$\mathrm{O}(n, 1) = \{g \in \mathrm{GL}(n+1, \mathbb{R}) \mid q(gx) = q(x) \ \forall x \in \mathbb{R}^{n+1}\}.$$

A general theorem of Witt states that $O(n, 1)$ acts transitively on all three kinds of level sets, except for the obvious exclusion of $0 \in \mathbb{R}^{n+1}$ in the case $a = 0$. We shall now restrict our attention to the upper sheet of $q^{-1}(-1)$ and denote it by $\mathbb{H}^n := \{x \in q^{-1}(-1) \mid x_{n+1} > 0\}$. To this end we introduce

$$\mathrm{SO}_0(n, 1) = \{g \in \mathrm{O}(n, 1) \mid g(\mathbb{H}^n) = \mathbb{H}^n \text{ and } \det g = 1\}.$$

The homomorphism $\mathrm{O}(n,1) \to \mathrm{Sym}(\pi_0(q^{-1}(-1))) \times \{-1,1\}$, given the permutation an element of $\mathrm{O}(n,1)$ induces on the two sheets of $q^{-1}(-1)$ and its determinant, is surjective and continuous for the discrete topology on the right hand side. Therefore, $\mathrm{SO}_0(n,1)$ is an open subgroup of index four in $\mathrm{O}(n,1)$. We now show that $\mathrm{SO}_0(n,1)$ acts transitively on $\mathbb{H}^n$. The stabilizer in $\mathrm{SO}_0(n,1)$ of e_{n+1} is given by

$$K := \left\{ \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \middle| A \in \mathrm{SO}(n) \right\}.$$

It acts as $\mathrm{SO}(n)$ in planes parallel to the plane given by $x_{n+1} = 0$: Given

$$k = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \in K$$

and $x = (x_1, \ldots, x_{n+1})^T \in \mathbb{H}^n$ we have

$$(x) = \begin{pmatrix} A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \\ x_{n+1} \end{pmatrix}.$$

In particular, there is $A \in \mathrm{SO}(n)$ such that

$$A \begin{pmatrix} x_1 & \vdots & x_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \sqrt{x_{n+1}^2 - 1} \end{pmatrix} \text{ and hence } k(x) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \sqrt{x_{n+1}^2 - 1} \\ 1 \end{pmatrix}$$

where $k \in \mathrm{SO}_0(n,1)$ is defined by A. In order to translate vertically, we introduce

$$A := \left\{ a_t := \begin{pmatrix} \mathrm{Id}_{n-1} & 0 \\ 0 & \begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix} \end{pmatrix} \middle| t \in \mathbb{R} \right\}$$

which is a one-parameter subgroup of $\mathrm{SO}_0(n,1)$. Indeed, it preserves the upper sheet of $q^{-1}(1)$, all its elements have determinant one and one computes $q(a_t x) = q(x)$ thanks to the identity $\cosh^2(t) - \sinh^2(t) \equiv 1$. Now,

$$a_t(e_{n+1}) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \sinh t \\ \cosh t \end{pmatrix}$$

which combined with the K-action shows transitivity of $\mathrm{SO}_0(n,1)$ on $\mathbb{H}^n$.

Corollary 1.65. The map $\mathrm{SO}_0(n,1)/K \cong K$, $gK \mapsto g(e_{n+1})$ is a diffeomorphism and $\mathrm{SO}_0(n,1)$ is connected.

Proof. The connectedness of $\mathrm{SO}_0(n,1)$ is due to the exercise that a total space of a fiber bundle with connected base and connected fiber is itself connected. □

Let us now compute the tangent space of $\mathbb{H}^n$ at $x \in \mathbb{H}^n$. To this end, let $b : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}$, $(x,y) \mapsto \sum_{i=1}^n x_i y_i - x_{n+1} y_{n+1}$ be the symmetric bilinear form associated to q. Then $D_x q(v) = 2b(x,v)$ and hence $\mathrm{T}_x \mathbb{H}^n = (\mathbb{R}\, x)^\perp$ where the orthogonal complement is taken with respect to b. Observe that since $q(x) = -1$ for all $x \in \mathbb{H}^n$ the space $\mathbb{R}\, x$ with form q is non-degenerate, hence the splitting

$$\mathbb{R}^{n+1} = \mathbb{R}\, x \oplus \mathrm{T}_x \mathbb{H}^n = \mathbb{R}\, x \oplus (\mathbb{R}\, x)^{\perp}.$$

Silvester's Inertia Theorem now implies that b restricted to $\mathrm{T}_x \mathbb{H}^n$ is positive definite. This constitutes an instance of a Riemannian metric as we shall see shortly. The space $\mathbb{H}^n$ is called real hyperbolic space of dimension n, and $\mathrm{SO}_0(n, 1)$ will act on it by isometries once we have properly defined the metric.

2. Riemannian Metrics, Covariant Derivative and Geodesics

2.1. **Definitions and Examples.** First of all, we define Riemannian metrics as encountered in the last section and prove their existence.

Definition 2.1. Let M be a smooth manifold. A *Riemannian metric* on M is a map g which to every point $p \in M$ associates a scalar product $g_p : \mathrm{T}_pM \times \mathrm{T}_pM \to \mathbb{R}$, satisfying the following smoothness condition: For any chart (U, φ) of M, the functions $U \to \mathbb{R},\ p \mapsto g_p(\partial_i p, \partial_j p)$ are smooth for all $1 \leq i, j \leq m$.

In the context of Definition 2.1 recall that

$$\partial_i(p)(f) = \frac{\partial(f \circ \varphi^{-1})}{\partial x_i}(\varphi(p))$$

for every $f \in C^\infty(M)$ and that $(\partial_1(p), \ldots, \partial_n(p))$ is a basis of T_pM for all $p \in M$.

Remark 2.2. It will be useful to have the bundle definition of Riemannian metrics as well: For a real vector space V, let $S^2(V^*)$ denote the vector space of all symmetric bilinear maps $V \times V \to \mathbb{R}$. Then one can define $S^2(\mathrm{T}^*M)$ as a smooth vector bundle of symmetric bilinear forms. That is, one introduces a manifold structure on

$$S^2(\mathrm{T}^*M) = \bigcup_{p \in M} \{p\} \times S^2((T_pM)^*)$$

as in the case of the tangent and cotangent bundle. A Riemannian metric g on M is then a smooth section $g : M \to S^2(\mathrm{T}^*M)$ such that $g_p := g(p) \gg 0$ for all $p \in M$. Notationally, given a Riemannian metric g on M, a chart (U, φ) and $u, v \in \mathrm{T}_pM$ with coordinates $u = \sum_{i=1}^m u_i \partial_i(p)$ and $v = \sum_{j=1}^m v_j \partial_j(p)$ respectively, we have

$$g_p(u, v) = \sum_{i,j} u_i v_j g_p(\partial_i(p), \partial_j(p))$$

where we abbreviate $g_{ij}(p) := g_p(\partial_i(p), \partial_j(p))$. Recall that $u_i = (dx_i)_p(u)$ and similarly for $v \in \mathrm{T}_pM$. Then in tensor notation, the bilinear form which maps (u, v) to $u_i v_j$ is $(dx_i)_p \otimes (dx_j)_p$. Thus, from the $S^2(\mathrm{T}^*M)$ viewpoint on Riemannian metrics, we have

$$g_p = \sum_{i,j} g_{ij}(p)(dx_i \otimes dx_j)_p.$$

The traditional (sloppy) way to refer to a Riemannian metric in local coordinates is $g = \sum_{i,j} g_{ij}\ dx_i dx_j$.

We now turn to the existence of Riemannian metrics which is unclear, given that we are asking for a highly non-zero section of a certain bundle and in view of what we have learned about e.g. vector fields on the two-dimensional sphere. Yet we have the following remarkable result whose proof is not even overly difficult.

Proposition 2.3. Every smooth manifold admits a Riemannian metric.

Proof. First, consider a single chart (U, φ) of M. Then we have an associated basis $(\partial_1(p), \ldots, \partial_m(p))$ of T_pM for all $p \in U$ whence for $u, v \in \mathrm{T}_pM$ we may define

$$g_p^U(u, v) := \sum_{i=1}^m u_i v_i,$$

employing the coordinates of u and v with respect to said basis. In other words, we set $g_p^U(\partial_i(p), \partial_j(p)) := \delta_{ij}$. This defines a Riemannian metric on U.

As to the whole manifold, let $(U_i, \varphi_i)_{i \in I}$ be a locally finite covering of M by charts and let $(f_i)_{i \in I}$ be a subordinate smooth partition of unity. For $p \in M$ now define

$$g_p := \sum_{i \in I} f_k(p) g_p^{U_i}.$$

Since for every $p \in M$ admits a neighbourhood V_p such that $\{i \in I \mid U_i \cap V_p \neq \emptyset\}$ is finite, g_p is a scalar product as a convex combination of scalar products. ☐

An alternative proof of Proposition 2.3 is to simply refer to the general version of Whitney's Embedding Theorem and restrict the Euclidean scalar product to the tangent spaces of the embedding.

In a sense, Whitney's theorem states that in order to study smooth manifolds one has to look no further than submanifolds of Euclidean space. An interesting question is whether this also holds for Riemannian manifold: Can every Riemannian manifold be embedded *isometrically* into some Euclidean space? The answer is yes and due to Nash.

Example 2.4. Arguably, the most immediate example of a Riemannian manifold is $\mathbb{R}^n$: For all $p \in \mathbb{R}^n$ and $u, v \in \mathrm{T}_p\mathbb{R}^n = \mathbb{R}^n$ set $g_p(u, v) = \langle u, v \rangle$ where $\langle -, - \rangle$ denotes the standard scalar product on $\mathbb{R}^n$.

Earlier on, we have seen that differential forms can be pulled back via general smooth maps and that vector fields can be pushed forward under some assumptions on the smooth maps. To the assumptions on has to put on a smooth map for it to allow pulling back a Riemannian metric are of intermediary nature: Let M and N be smooth manifolds and let $f : M \to N$ be a smooth immersion. If h is a Riemannian metric on N then

$$g_p(u, v) := h_{f(p)}(D_p f(u), D_p f(v))$$

where $p \in M$ and $u, v \in \mathrm{T}_pM$ defines a Riemannian metric on g, denoted $f^*(h)$.

Definition 2.5. Let (M, g) and (N, h) be Riemannian manifolds and let $f : M \to N$ be a smooth map. Then f is an *isometry* if f is a diffeomorphism and $g = f^*(h)$.

Given a Riemannian manifold (M, g), the set $\mathrm{Iso}(M, g)$ of isometries of M is a subgroup of the group of diffeomorphisms of M. Whereas the latter is infinite-dimensional from any point of view, $\mathrm{Iso}(M, g)$ is a Lie group by a result of Myers and Steenrod. Later on we will look at the particularly intriguing case in which $\mathrm{Iso}(M, g)$ acts transitively on M.

One of the points of having a Riemannian metric is to be able to measure the length of certain curves.

Definition 2.6. Let (M, g) be a Riemannian manifold and let $c : [a, b] \to M$ be a C^1-curve. The *length* $l(c)$ of c is

$$l(c) := \int_a^b \sqrt{g_{c(t)}(c'(t), c'(t))}\, dt.$$

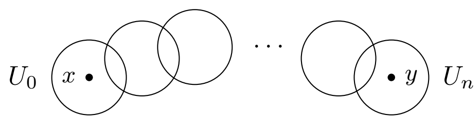
It is useful to extend Definition 2.6 to piecewise C^1-curves: If $c : [a,b] \to M$ is C^1 on $[a_i, a_{i+1}]$ for all $i \in [0, n-1]$ where $a_0 = a < a_1 < \cdots < a_n = b$ then we set $l(c) := \sum_{i=0}^{n-1} l(c|_{[a_i,a_{i+1}]})$.

Note that we are defining the length of a curve using its instant velocities which is quite the other way around as in calculus where there are length and time first, then their quotient. We will see more instances of this reverse engineering. Anyway, one shows that the length of a piecewise C^1-curve does not depend on the choice of parametrization. Also, one can always reparametrize such a curve by arc length in which case $\|c'(t)\| = 1$. One of the first problems in Riemannian geometry is to find curves of shortest length between given points. We will later on characterize these as having constant speed and "zero acceleration". The term "acceleration" however requires clarification as the notion of "second derivative of a curve" is just not there. This will be remedied by introducing *connections* which provide a preferred way of transporting tangent vectors along a curve. Whereas there are plenty connections in general, Riemannian manifolds come with a natural one, the *Levi-Civita connection*.

Now, given a connected Riemannian manifold M and $x, y \in M$ we define the *distance* between a and b by

$$d(x,y) := \inf\{l(c) \mid c : [a,b] \to M \text{ piecewise } C^1,\ c(a) = x,\ c(b) = y\}.$$

Note that in a connected Riemannian manifold as above, there always is a piecewise C^1-curve between any two given points: Find a sequence of coordinate charts $(U_i, \varphi_i)_{i=0}^n$ with $x \in U_0$, $y \in U_n$ and $U_i \cap U_{i+1} \neq \emptyset$.

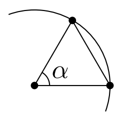


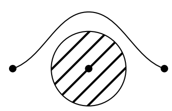
Proposition 2.7. Let (M, g) be a connected Riemannian manifold and define the map $d : M \times M \to \mathbb{R}$ by

$$d(x,y) := \inf\{l(c) \mid c : [a,b] \to M \text{ piecewise } C^1,\ c(a) = x,\ c(b) = y\}.$$

Then d is a metric on M inducing the present topology.

Although this does not present much difficulties we skip the proof and move on to examples of Riemannian manifolds and construction methods.





Example 2.8. This example is to illustrate that for a submanifold M of $\mathbb{R}^n$, the distance function induced from the restricted scalar product is generally quite different from the ambient distance function: For instance, consider the sphere $S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$. Then the distance on S_n between two points x and y on the sphere at an angle α is α whereas their distance in Euclidean space is $2(1 - \cos\alpha)$. Also, erasing a neighbourhood of $0 \in \mathbb{R}^2$ naturally results in an altered metric.

In particular, note that given a metric space one can perfectly restrict the metric to a subset and thereby produce a new metric space, this is not what happens when one restricts a Riemannian metric to a submanifold.

Example 2.9. The hyperbolic spaces $\mathbb{H}^n$ introduced in Example 1.64 provide more key Riemannian manifolds. Retain the notation of said example. Given $x \in \mathbb{H}^n$, set $h_x(u,v) := b(u,v)$ for all $u, v \in \mathrm{T}_x\mathbb{H}^n$. Then h is a Riemannian metric on $\mathbb{H}^n$. Observe that for every $g \in \mathrm{SO}_0(n,1)$ and $x \in \mathbb{H}^n$, the map $D_x g$ sends h_x to $h_{g(x)}$, i.e. $\mathrm{SO}_0(n,1)$ acts by isometries. This yields an injective group homomorphism

$SO_0(n,1) \to \mathrm{Iso}(\mathbb{H}^n, h)$. Later on, we shall see that this homomorphism is almost surjective. Another model of hyperbolic space $\mathbb{H}^n$ via the diffeomorphism

$$f : D^n := \{y \in \mathbb{R}^n \mid \|y\| < 1\} \to \mathbb{H}^n,\ y \mapsto \left(\frac{2y_i}{1-\|y\|^2}, \frac{1+\|y\|^2}{1-\|y\|^2} \right)$$

One verifies that f indeed ranges in $\mathbb{H}^n$ by computing $q(f(y))$ and that it admits a smooth inverse. Furthermore, one obtains

$$g := f^*(h) = 4\sum_{i=1}^{n} \frac{(dy_i)^2}{(1-\|y\|^2)},\ \text{i.e. } g_y(u,v) = \frac{4\langle u, v\rangle}{(1-\|y\|^2)^2}.$$

In this model the boundary S^{n-1} of D^n is infinitely far away from the origin: Indeed, for $0 \le r < 1$ one computes

$$l_g([0, re_1]) = 4\int_0^r \frac{dt}{1-t^2} = \ln\left(\frac{1+r}{1-r}\right).$$

Another interesting observation is that whereas a random walk starting at the origin in $\mathbb{E}^2$ will return to a neighbourhood of the origin infinitely many times. In the disk model of $\mathbb{H}^2$, however, random walks do not come back, due to the expanding space: Whereas the area of a ball of radius R in $\mathbb{E}^2$ behaves like R^2, its behaviour with respect to the hyperbolic metric is exponential in R. Also, whereas the area of a Euclidean annulus of width R behaves like R, a hyperbolic annulus with larger radius equal to one already contains a large fraction of the entire area.

For later use, we now record two formal definitions that allow us to construct new Riemannian manifolds out of old.

Definition 2.10. Let (M, g) and (N, h) be a Riemannian manifolds. Then (N, h) is a *Riemannian submanifold* of (M, g) if

(i) N is a submanifold of M, and
(ii) $i^*(g) = h$ where $i : N \to M$ is the canonical injection.

Definition 2.11. Let (M, g) and (N, h) be Riemannian manifolds. Then $M \times N$ is a Riemannian manifold with metric $g \times h$ defined by

$$(g \times h)_{(x,y)}(u, v) = g_x(u_1, v_1) + h_y(u_2, v_2)$$

where $u = (u_1, u_2)$ and $v = (v_1, v_2)$ with respect to $\mathrm{T}_{(x,y)}(M \times N) = \mathrm{T}_x M \times \mathrm{T}_y N$.

It is also important to know how Riemannian metric behave with respect to covering maps.

Definition 2.12. Let (M, g) and (N, h) be Riemannian manifolds and let $p : N \to M$ be a smooth map. Then p is a *Riemannian covering* if

(i) p is a smooth covering, and
(ii) $h = p^*g$.

In particular, a Riemannian covering is a local isometry. It is plain that if $p : N \to M$ is a smooth covering of smooth manifolds and g is a Riemannian metric on M then p is a Riemannian covering when N is equipped with the Riemannian metric $h = p^*(g)$.

We now turn to the question under which circumstances a Riemannian metric on N can be pushed down to M via p.

Proposition 2.13. Let (N, h) be a Riemannian manifold and let Γ be a group. Further, let $\Gamma \times N \to N$ be an action of Γ on N which is free, properly discontinuously, and by isometries. Then there is a unique Riemannian metric g on the quotient manifold $M := \Gamma\backslash N$ such that $p : N \to M$ is a Riemannian covering.

Proof. Let $x \in M$ and suppose y and y' are elements of $p^{-1}(x)$. Then there is an isometry $\gamma \in \Gamma$ with $\gamma y = y'$ and the diagram

$$\begin{array}{ccc} N & \xrightarrow{\gamma} & N \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p} & \\ & M & \end{array}$$

commutes. So does the diagram obtained by taking derivates.

$$\begin{array}{ccc} \mathrm{T}_x N & \xrightarrow{D_y\gamma} & \mathrm{T}_{y'} N \\ & {\scriptstyle D_y p}\searrow \quad \swarrow{\scriptstyle D_{y'} p} & \\ & \mathrm{x}M & \end{array}$$

Since p is a covering map, both $D_y p$ and $D_{y'} p$ are vector space isomorphisms. We use $D_y p$ to introduce a scalar product on $\mathrm{T}_x M$ by $g_x := ((D_y p)^{-1})^*(h_y)$. Observe that since γ is an isometry we have $(D_y \gamma)^*(h_{y'}) = h_y$. Therefore

$$\begin{aligned} g_x = ((D_y p)^{-1})^*(h_y) &= ((D_y p)^{-1})^*(D_y \gamma)^*(h_{y'}) \\ &= (D_y \gamma \circ (D_y p)^{-1})^*(h_{y'}) = ((D_{y'} p)^{-1})^*(h_{y'}) \end{aligned}$$

which shows that g_x is well-defined. The smoothness of the resulting Riemannian metric is due to the fact that p is a local diffeomorphism. This way, p becomes a Riemannian covering by construction. □

Example 2.14. We equip $\mathbb{R}^n$ with the Riemannian metric coming from the standard scalar product $\langle x, y \rangle_p := \sum_{i=1}^n x_i y_i$, denoted can. One can then show that the Riemannian distance of $x, y \in \mathbb{R}^n$ given by

$$d(x, y) := \inf\{l(c) \mid c : [a, b] \to M \text{ piecewise } C^1,\ c(a) = x,\ c(b) = y\}.$$

coincides with the Euclidean distance of x and y given by $\|x-y\| = \sqrt{\langle x - y, x - y\rangle}$. As a hint, consider the smooth, distance non-increasing projection onto the straight line connecting two points or go back to the definition of the Riemann integral and the mean value theorem.

Proposition 2.15. Every Riemannian isometry g of $(\mathbb{R}^n, \mathrm{can})$ is given by $g : v \mapsto Rv + a$ for some $R \in \mathrm{O}(n)$ and $a \in \mathbb{R}^n$.

Sketch of Proof. By Example 2.14, an isometry $g \in \mathrm{Iso}(\mathbb{R}^n)$ is a bijection which satisfies $\|g(x) - g(y)\| = \|x - y\|$ for all $x, y \in \mathbb{R}^n$. Now one can invoke the Mazur-Ulam theorem.

Theorem 2.16 (Mazur-Ulam). Let V be a normed vector space over $\mathbb{R}$ and let $T : V \to V$ be a bijection satisfying $\|g(x) - g(y)\| = \|x - y\|$ for all $x, y \in V$. Then T is of the form $T(x) = Ax + b$ where $a : V \to V$ is linear.

By this theorem, we have $g(v) = R(v) + a$ for all $v \in \mathbb{R}^n$ where R is linear and $\|R(x) - R(y)\| = \|x - y\|$. By the parallelogram law, this implies $R \in \mathrm{O}(n)$: $\langle x, y\rangle = (\|x + y\|^2 - \|x - y\|^2)/4$.

We now examine the group law on $\mathrm{Iso}(\mathbb{R}^n, \mathrm{can})$: Every $g \in \mathrm{Iso}(\mathbb{R}^n)$ is represented by a pair (R, a) in the set-theoretic cartesian product $\mathrm{O}(n) \times \mathbb{R}^n$. Given isometries $g_1 = (R_1, a_1)$ and $g_2 = (R_2, a_2)$ we compute

$$g_1 g_2(v) = g_1(R_2 v + a_2) = R_1 R_2 v + R_1 a_2 + a_1$$

and thus $g_1g_2 = (R_1R_2, R_1a_2 + a_1)$. Thus, as a group, $\mathrm{Iso}(\mathbb{R}^n, \mathrm{can})$ is isomorphic to the semidirect product $\mathrm{O}(n) \ltimes \mathbb{R}^n$. The map $r : \mathrm{O}(n) \ltimes \mathbb{R}^n \to \mathrm{O}(n)$ is a homomorphism whose kernel is the group of pure translations. Given $a \in \mathbb{R}^n$ we write $T_a(v) = v + a$.

Riemannian coverings $p : \mathbb{R}^n \to M$ are given by subgroups $\Gamma\mathrm{Iso}(\mathbb{R}^n)$ which act freely and properly discontinuously on $\mathbb{R}^n$. In this context, we have the following.

Definition 2.17. A subgroup $\Gamma \leq \mathrm{Iso}(\mathbb{R}^n, \mathrm{can})$ is *crystallographic* if it acts properly discontinuously and $\Gamma\backslash\,\mathbb{R}^n$ is compact. It is *Bieberbach* if it is crystallographic and acts freely.

In particular, a Bieberbach group $\Gamma \leq \mathrm{Iso}(\mathbb{R}^n, \mathrm{can})$ defines a Riemannian covering $p_\Gamma : \mathbb{R}^n \to \Gamma\backslash\,\mathbb{R}^n$ where $\Gamma\backslash\,\mathbb{R}^n$ is a compact manifold.

Example 2.18. Let P denote the subgroup of $\mathrm{O}(n)$ consisting of permutation matrices and set $\Gamma := \{(R, \gamma) \mid R \in P,\ \gamma \in \mathbb{Z}^n\}$ is crystallographic but not Bieberbach.

Bieberbach groups are more difficult to construct. The trivial ones are the content of the following proposition.

Proposition 2.19. Let $(a_1, \dots, a_n)$ be a basis of $\mathbb{R}^n$ and $\Gamma = \{T_\gamma \mid \gamma \in \mathbb{Z}\,a_1 + \cdots \mathbb{Z}\,a_n\}$. Then Γ is Bieberbach and $\Gamma\backslash\,\mathbb{R}^n$ is diffeomorphic to $\mathbb{T}^n = S^1 \times \cdots \times S^1$. Furthermore, the Riemannian manifolds $\Gamma\backslash\,\mathbb{R}^n$ and $\Gamma'\backslash\,\mathbb{R}^n$ are isomorphic if and only if there is $R \in \mathrm{O}(n)$ with $R(\mathbb{Z}\,a_1 + \cdots + \mathbb{Z}\,a_n) = \mathbb{Z}\,a'_1 + \cdots + \mathbb{Z}\,a'_n$ where $\Gamma = \{T_\gamma \mid \gamma \in \mathbb{Z}\,a_1 + \cdots \mathbb{Z}\,a_n\}$ and $\Gamma' = \{T_{\gamma'} \mid \gamma' \in \mathbb{Z}\,a'_1 + \cdots \mathbb{Z}\,a'_n\}$.

Proof. Consider the map $e : \mathbb{R}^n \to \mathbb{T}^n$ given by $x \mapsto \left(e^{2\pi i x_1}, \dots, e^{2\pi i x_n}\right)$ where $x = (x_1, \dots, x_n)^T \in \mathbb{R}^n$. The map e is a smooth covering and we have $e(x) = e(y)$ if and only if $x - y \in \mathbb{Z}\,a_1 + \cdots + \mathbb{Z}\,a_n$, that is if and only if $T_\gamma(x) = y$ for some $T_\gamma \in \Gamma$. This implies that $e : \Gamma\backslash\,\mathbb{R}^n \to \mathbb{T}^n$, being a local diffeomorphism and bijective, is a diffeomorphism.

Assume now that $f : \Gamma\backslash\,\mathbb{R}^n \to \Gamma'\backslash\,\mathbb{R}^n$ is an isometry. Since p_Γ and $p_{\Gamma'}$ are covering maps and $\mathbb{R}^n$ is simply connected, there exists a diffeomorphism $\widetilde{f} : \mathbb{R}^n \to \mathbb{R}^n$ such that the following diagram commutes:

$$\begin{array}{ccc} \Gamma\backslash\,\mathbb{R}^n & \xrightarrow{\ f\ } & \Gamma'\backslash\,\mathbb{R}^n \\ {\scriptstyle p_\Gamma}\uparrow & & \uparrow{\scriptstyle p_{\Gamma'}} \\ \mathbb{R}^n & \xrightarrow[\ \widetilde{f}\ ]{} & \mathbb{R}^n . \end{array}$$

Now observe that $\widetilde{f}$ is an isometry: Let g_Γ and $g_{\Gamma'}$ denote the Riemannian metrics on $\Gamma\backslash\,\mathbb{R}^n$ and $\Gamma'\backslash\,\mathbb{R}^n$ respectively for which p_Γ and $p_{\Gamma'}$ are Riemannian coverings. Furthermore, let g_{can} denote the canonical Riemannian metric on $\mathbb{R}^n$. Then

$$\begin{aligned} \widetilde{f}^*(g_{\mathrm{can}}) = \widetilde{f}^* p_{\Gamma'}^*(g_{\Gamma'}) &= (p_{\Gamma'} \circ \widetilde{f})^*(g_{\Gamma'}) \\ &= (f \circ p_\Gamma)^*(g_{\Gamma'}) = p_\Gamma^* f^*(g_{\Gamma'}) = p_\Gamma^*(g_\Gamma) = g_{\mathrm{can}}. \end{aligned}$$

By the above we conclude that $\widetilde{f} = (R, a)$ for some $R \in \mathrm{O}(n)$ and $a \in \mathbb{R}^n$. From this one deduces that Γ and Γ' are conjugate as subgroups of $\mathrm{Iso}(\mathbb{R}^n)$: $\widetilde{f}\Gamma\widetilde{f}^{-1} = \Gamma'$. This implies $R(\mathbb{Z}\,a_1 + \cdots + \mathbb{Z}\,a_n) = \mathbb{Z}\,a'_1 + \cdots + \mathbb{Z}\,a'_n$. □

Remark 2.20. The Riemannian manifolds described in Proposition 2.19 are called *flat tori* because they are diffeomorphic to a torus and flat from a curvature point of view as we shall see later. We now ask whether there is a "reasonable space" parametrizing the set of flat tori up to isometry: Let R be the set of all subgroups of $\mathbb{R}^n$ of the form $\mathbb{Z}\,a_1 + \cdots + \mathbb{Z}\,a_n$ where $(a_1, \dots, a_n)$ is a basis of $\mathbb{R}^n$. Clearly, $\mathrm{GL}(n, \mathbb{R})$

acts on R: Given $\Lambda = \mathbb{Z}\,a_1 + \cdots + \mathbb{Z}\,a_n$ and $g \in \mathrm{GL}(n, \mathbb{R}$ we have $g(\Lambda) = \mathbb{Z}\,ga_1 + \cdots + \mathbb{Z}\,ga_n$ where $(ga_1, \ldots, ga_n)$ is again a basis of $\mathbb{R}^n$. This action is transitive: Fix $\Lambda_0 = \mathbb{Z}\,e_1 + \cdots + \mathbb{Z}\,e_n$. Given $\Lambda = \mathbb{Z}\,a_1 + \cdots + \mathbb{Z}\,a_n$ set $g = (a_1, \ldots, a_n) \in \mathrm{GL}(n, \mathbb{R})$. Then $g(\Lambda_0) = \Lambda$. The stabilizer in $\mathrm{GL}(n, \mathbb{R})$ of Λ_0 is given by

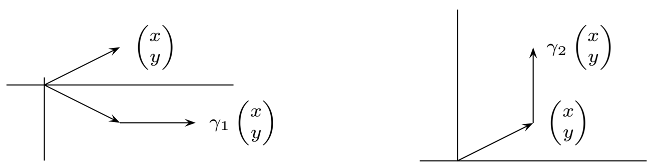
$$\begin{aligned}&\{g \in \mathrm{GL}(n, \mathbb{R}) \mid g(\Lambda_0) = \Lambda_0\}\\ &\quad = \{g \in \mathrm{GL}(n, \mathbb{R}) \mid \forall i, j \in \{1, \ldots, n\} :\ g_{ij} \in \mathbb{Z},\ g_{ij}^{-1} \in \mathbb{Z}\}\\ &\quad = \{g \in \mathrm{GL}(n, \mathbb{R}) \mid \forall i, j \in \{1, \ldots, n\} :\ g_{ij} \in \mathbb{Z} \text{ and } \det g \in \{\pm 1\}\}.\end{aligned}$$

Hence $R \cong \mathrm{GL}(n, \mathbb{R})/\mathrm{GL}(n, \mathbb{Z})$ as sets. Observe that $\mathrm{GL}(n, \mathbb{Z})$ is a discrete subgroup of the Lie group $\mathrm{GL}(n, \mathbb{R})$ whence $\mathrm{GL}(n, \mathbb{R})/\mathrm{GL}(n, \mathbb{Z})$ has a canonical manifold structure. Taking into account isometry classes one sees that at the end of the day, flat tori are parametrized by the double quotient $\mathrm{O}(n)\backslash \mathrm{GL}(n, \mathbb{R})/\mathrm{GL}(n, \mathbb{Z})$ where $\mathrm{GL}(n, \mathbb{Z})$ acts on $\mathrm{O}(n)\backslash \mathrm{GL}(n, \mathbb{R})$ on the right because $\mathrm{O}(n)$ is compact. Whereas the above double quotient has a nice structure, equally natural questions lead to double quotients where the left hand subgroup is non-compact, resulting in the fact that the double quotient is not even a standard Borel space.

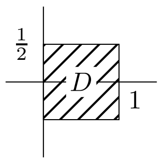
A less obvious Bieberbach group is $\Gamma := \langle \gamma_1, \gamma_2 \rangle \leq \mathrm{Iso}(\mathbb{R}^2)$ where

$$\gamma_1 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \gamma_2 = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We claim that Γ is a Bieberbach group, i.e. it acts properly discontinuously, freely and with compact quotient on $\mathbb{R}^2$. Geometrically, we have



To get a geometric idea of the quotient, consider the domain

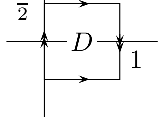


$$D = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \,\middle|\, 0 \leq x \leq 1,\ |y| \leq \frac{1}{2} \right\}$$

Any $v \in \mathbb{R}^2$ is Γ-quivalent to a point in D: Given $v = (x, y)^T$, applying an appropriate power k of γ_2 yields $\gamma_2^k(x, y)^T = (x', y')$ with $|y'| \leq \frac{1}{2}$. Then, consider

$$\gamma_1^l \gamma_2^k (x, y)^T = \gamma_1^l (x', y')^T = (x' + l, (-1)^l y')$$

so that $0 \leq x' + l \leq 1$ for appropriate l while $|(-1)^l y'| = |y'| \leq 1/2$. One easily determines the identifications that Γ induces on the boundary of D, namely



The quotient $\Gamma\backslash \mathbb{R}^2$ is termed *Klein bottle*. Note that Γ contains the purely translational subgroup $\Gamma' := \{T_\gamma \mid \gamma \in 2\,\mathbb{Z}\,e_1 + \mathbb{Z}\,e_2\}$ and that the torus $\Gamma'\backslash \mathbb{R}^2$ projects with degree two onto $\Gamma\backslash \mathbb{R}^2$. It is a good exercise to visualize this projection. Later on, we will study the Riemannian geometries on these examples.

An example of a Bieberbach group in three dimensions is the *Hantsche-Wendt* group $\Gamma := \langle \gamma_1, \gamma_2 \rangle \leq \mathrm{Iso}(\mathbb{R}^3)$ where

$$\gamma_1 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -x \\ -y \\ z \end{pmatrix} + \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \end{pmatrix} \quad \text{and} \quad \gamma_2 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ -y \\ -z \end{pmatrix} + \begin{pmatrix} 1/2 \\ 0 \\ 0 \end{pmatrix}.$$

Interestingly, $\Gamma \backslash \mathbb{R}^3$ and S^3 have the same Betti numbers but are not diffeomorphic: They can be distinguished with integral homology for instance. The following theorem of Bieberbach shows that it is always a good idea to look for a purely translational subgroup in order to prove cocompactness. Recall that $r : \mathrm{Iso}(\mathbb{R}^n) \to \mathrm{O}(n)$ denotes the map which to an isometry associates its linear part.

Theorem 2.21 (Bieberbach)*.* Let $\Gamma \backslash \mathrm{Iso}(\mathbb{R}^n)$ be a crystallographic grup. Then

(i) $r(\Gamma)$ is a finite subgroup of $\mathrm{O}(n)$,
(ii) $\Gamma \cap \{T_a \mid a \in \mathbb{R}^n\}$ is Bieberbach, and
(iii) Γ acts freely on $\mathbb{R}^n$ if and only if it is torsion-free.

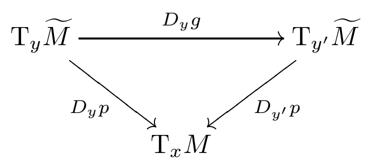
The proof of Theorem 2.21 requires several good ideas. We refer the reader to [Cha12] and the classic [Wol11].

Riemannian Submersions. We have seen how Riemannian metrics behave with respect to coverings. Now, we look at the more general case of submersions: Let $p : M \to N$ be a submersion, i.e. $D_x p : \mathrm{T}_x M \to \mathrm{T}_{p(x)} N$ is surjective for all $x \in M$, and let g be a Riemannian metric on M. Then we can define the *horizontal subspace* $H_x := (\ker D_x p)^\perp$ where the orthogonal complement is taken with respect to the Riemannian metric. Clearly, $D_x p|_{H_x} : H_x \to \mathrm{T}_{p(x)} N$ is an isomorphism.

Definition 2.22*.* Retain the above notation. The submersion p is *Riemannian* if the map $D_x p|_{H_x} : (H_x, g|_{H_x}) \to (\mathrm{T}_{p(x)} N, h_{p(x)})$ is an isometry.

Proposition 2.23*.* Let $(\widetilde{M}, \widetilde{g})$ be a Riemannian manifold and let $G \times \widetilde{M} \to \widetilde{M}$ be a free, proper Lie group action by isometries of G on $\widetilde{M}$. Then there is a unique Riemannian metric g on the quotient $M := G \backslash \widetilde{M}$ such that $p : \widetilde{M} \to M$ is a Riemannian submersion.

Proof. The proof is quite analogous to the proof of Proposition 2.13: Let $x \in M$ and $y, y' \in p^{-1}(x)$. Also, let $g \in G$ be such that $g(y) = y'$. Then the following diagram commutes:



In particular, $D_y g(\ker D_y p) = \ker D_{y'} p$. Since $(D_y g)^*(\widetilde{g}_{y'})$ we get $D_y g(H_y) = H_{y'}$ where $H_y = (\ker D_y p)^\perp$ and $H_{y'} = (\ker D_y p)^\perp$. Now define

$$g_x = ((D_y p|_{H_y})^{-1})^*(\widetilde{g}_y).$$

Then the above shows that g_x is independent of the choice of $y \in p^{-1}(x)$. □

Example 2.24*.* Recall that $\mathbb{P}^n(\mathbb{C}) = \mathbb{C}^* \setminus \mathbb{C}^{n+1} - \{0\}$ has a manifold structure with charts arising from affine hyperplanes in $\mathbb{C}^{n+1}$. In fact, it is even a complex analytic manifold. Whereas in the real case, the projection $S^n \to (\mathbb{Z}/2\mathbb{Z}) \backslash S^n = \mathbb{P}^n(\mathbb{R})$ is a covering, we obtain a non-discrete fibration in the complex case: Consider $S^{2n+1} = \{z \in \mathbb{C}^{n+1} \mid \sum_{i=1}^{n+1} |z_i|^2 = 1\}$ and the inclusion $S^{2n+1} \to \mathbb{C}^{n+1} \setminus \{0\}$. The Lie group $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ acts on S^{2n+1} by $\zeta(z_1, \ldots, z_{n+1}) = (\zeta z_1, \ldots, \zeta z_{n+1})$. This action is free and proper; in fact any smooth action of a compact Lie group is proper. Now, the inclusion $S^{2n+1} \to \mathbb{C}^{n+1} \setminus \{0\}$ induces a surjective submersion $p : S^{2n+1} \to \mathbb{P}^n(\mathbb{C})$ and one can show that it induces a diffeomorphism

$$S^1 \backslash S^{2n+1} \to \mathbb{P}^n(\mathbb{C}).$$

Let $(z, w) = \sum_{i=1}^{n+1} z_i \overline{w}_i$ be the unitary scalar product on $\mathbb{C}^{n+1}$. Then the Euclidean scalar product on the real vector space $\mathbb{C}^{n+1}$ is given by $\langle u, w\rangle = \mathrm{Re}(u, w)$. We now equip $\mathbb{C}^{n+1}$ with the standard Riemannian metric coming from $\langle -, -\rangle$ and $S^{2n+1} \subseteq \mathbb{C}^{n+1}$ with the induced Riemannian metric. Then for $z \in S^{2n+1}$ we have

$$\mathrm{T}_z S^{2n+1} = \{u \in \mathbb{C}^{n+1} \mid \langle z, u\rangle = 0\} = \{u \in \mathbb{C}^{n+1} \mid \mathrm{Re}(z, u) = 0\}.$$

and the orthogonal decomposition $\mathbb{C}^{n+1} = \mathbb{R}\, z \oplus \mathrm{T}_z S^{2n+1}$ where both constituents of the right hand side are real vector spaces. The S^1-orbit through $z \in S^{2n+1}$ can be parametrized by $c : \mathbb{R} \to S^{2n+1}$, $t \mapsto e^{it} \cdot z$. Hence $\ker D_z p = \mathbb{R} \cdot i \cdot z$ where $p : S^{2n+1} \to S^1 \backslash S^{2n+1}$. Using $H_z \subseteq \mathrm{T}_z S^{2n+1}$ we have the following orthogonal decomposition by definition:

$$\mathrm{T}_z S^{2n+1} = \mathbb{R}\, iz \oplus H_z.$$

On $\mathbb{P}^n\,\mathbb{C}$ we put the Riemannian metric coming from S^{2n+1} or rather the restriction to H_z. Thus, putting everything together, we have an orthogonal decomposition

$$\mathbb{C}^{n+1} = \underbrace{\mathbb{R}\, z + \mathbb{R}\, iz}_{=\mathbb{C}\, z} + H_z$$

where H_z is of real codimension one in $\mathrm{T}_z S^{2n+1}$. Hence H_z is in fact orthogonal to $\mathbb{C}_z$ for the unitary scalar product and hence a complex vector space. This is a good example to keep in mind for a submersion with interesting additional structure.

In the context of the above example it is worth mentioning that one can do differential geometry over the complex numbers but that things become more rigid. For instance, there is no analogue of Whitney's embedding theorem or embeddings into projective space. In fact, complex submanifolds of complex projective space are very special.

Lie Groups. As usual, things work particularly well in the context of Lie groups.

Example 2.25. Let G be a Lie group and let $\langle -, -\rangle_e$ be a scalar product on $\mathrm{T}_e G$. Then we can define $\langle u, v\rangle_g = \langle D_g L_{g^{-1}}(u), D_g L_{g^{-1}}(v)\rangle_e$ for all $u, v \in \mathrm{T}_g G$, producing a left-invariant Riemannian metric $\widetilde{g}$ on G. In particular, G can be viewed as a subgroup of $\mathrm{Iso}(G, \widetilde{g})$.

One might as well construct a right-invariant Riemannian metric on a Lie group. In general, however, there need not be one which is bi-invariant.

Proposition 2.26. Let G be a Lie group. Then the following are equivalent.

(i) The Lie group G admits a bi-invariant Riemannian metric.
(ii) The image $\mathrm{Ad}(G) \subseteq \mathrm{GL}(\mathfrak{g})$ is contained in a compact subgroup.

Corollary 2.27. Let G be a compact Lie group. Then G admits a bi-invariant Riemannian metric.

2.2. **Covariant Derivative.** This section truly kicks off the theory of Riemannian geometry by introducing the crucial notion of *connection*, which was only understood in the 1920's by very few people, including É. Cartan. Nevertheless, it leads to the Riemannian curvature tensor which was introduced much earlier, namely in 1853, by Riemann.

The problem we study is the following: Let X and Y be smooth vector fields on $\mathbb{R}^n$; later on we shall be interested in general manifolds, of course. Is there a natural notion of taking the derivative of Y with respect to X, that is the "variation of Y

from the point of view of X"? One possible way to deal with this is to consider the integral curve $c_p : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ of X passing through $p \in M$ at time $t = 0$,

$$\begin{cases} c_p'(t) = X(c_p(t)) \\ c_p(0) = p \end{cases},$$

and to look at the vectors $Y(c_p(t)) \in \mathrm{T}_{c_p(t)} \mathbb{R}^n$. We would like to define the first order variation of the map $t \mapsto Y(c_p(t))$ near $t = 0$. If we had started out with a general manifold we would be stuck at this point because the different tangent spaces don't know about each other, that is, we have no means to compare tangent vectors at different points. In $\mathbb{R}^n$, however, we can utilize the translations to establish a privileged identification $\mathrm{T}_{c_p(t)} \mathbb{R}^n \to \mathrm{T}_p \mathbb{R}^n$. Using this identification we consider $Y(c_p(t))$ to be an element of $\mathrm{T}_p \mathbb{R}^n$ and define

$$(\nabla_X Y)(p) := \lim_{t \to 0} \frac{Y(c_p(t)) - Y(p)}{t}$$

Writing $Y = (Y_1, \ldots, Y_n)$ and $X = (X_1, \ldots, X_n)$ one verifies that

$$(\nabla_X Y)(p) = \left(\ldots, \sum_{i=1}^{n} \frac{\partial Y_j(p)}{\partial x_i} X_i(p), \ldots \right),$$

its merely a directional derivative. The map $\nabla : \Gamma(\mathrm{T}\,\mathbb{R}^n) \times \Gamma(\mathrm{T}\,\mathbb{R}^n) \to \Gamma(\mathrm{T}\,\mathbb{R}^n)$ satisfies the following formal properties.

(i) $\nabla_X Y - \nabla_Y X = [X, Y]$ for all $X, Y \in \Gamma(\mathrm{T}\,\mathbb{R}^n)$.
(ii) $\nabla_{fX} Y = f \nabla_X Y$ for all $X \in \Gamma(\mathrm{T}\,\mathbb{R}^n)$ and all $f \in C^\infty(\mathbb{R}^n)$.
(iii) $\nabla_X (fY) = (Xf)Y + f \nabla_X Y$ for all $X, Y \in \Gamma(\mathrm{T}\,\mathbb{R}^n)$ and $f \in C^\infty(\mathbb{R}^n)$.

Using these properties, we now define connections in general. This will lead to *parallel transport* which allows to identify tangent spaces at distinct points of a manifold depending on a path connecting them, taking curvature into account. The fact that our identification in $\mathbb{R}^n$ does not depend on a path connecting given points resembles the vanishing of curvature.

Definition 2.28. Let M be a smooth manifold. A *connection on* M is an $\mathbb{R}$-linear map $\nabla : \Gamma(\mathrm{T}M) \times \Gamma(\mathrm{T}M) \to \Gamma(\mathrm{T}M)$ which satisfies the following properties for all $X, Y \in \Gamma(\mathrm{T}M)$ and $f \in C^\infty(M)$:

(i) $\nabla_{fX} Y = f \nabla_X Y$,
(ii) $\nabla_X (fY) = (Xf)Y + f \nabla_X Y$, and
(iii) $\nabla_X Y - \nabla_Y X = [X, Y]$.

Note that defining ∇ to be the zero map does not yield a connection due to the second property. Hence existence of connections is not completely obvious. However, if ∇^1 and ∇^2 are connections on a manifold M then so is $g\nabla^1 + (1-g)\nabla^2$ for every $g \in C^\infty(M)$. Therefore, using a partition of unity to patch together connections defined on charts, one shows that every smooth manifolds admits many different connections and thus many different notions of acceleration which mathematicians had a hard time to accept.

Remark 2.29. Let ∇ be a connection on a smooth manifold M. Then the first condition implies that for given $Y \in \Gamma(\mathrm{T}M)$ the vector $(\nabla_X Y)(p)$ $(p \in M)$ only depends on $X(p)$ rather than the values of X in a neighbourhood of $p \in M$: Indeed, let (U, φ) be a chart of M at p and recall that for every $q \in U$ we obtain a basis $(\partial_1(q), \ldots, \partial_m(q))$ of $\mathrm{T}_q M$. Consider $\partial_i \in \Gamma(\mathrm{T}U)$ $(i \in \{1, \ldots, m\})$ as a smooth vector field. Then the local expression of X on U takes the form

$X(q) = \sum_{i=1}^m X_i(q)\partial_i(q)$. In vector field notation: $X = \sum_{i=1}^m X_i\partial_i$. Therefore

$$\nabla_X Y = \nabla_{\sum_{i=1}^m X_i\partial_i} Y = \sum_{i=1}^m X_i \nabla_{\partial_i} Y$$

and hence $(\nabla_X Y)(p) = \sum_{i=1}^m X_i(p)(\nabla_{\partial_i} Y)(p)$. It follows that for $X, X' \in \Gamma(\mathrm{T}M)$ with $X(p) = X'(p)$ we have $X_i(p) = X_i'(p)$ and hence $(\nabla_X Y)(p) = (\nabla_{X'} Y)(p)$.

The next theorem probably deserves to be called the fundamental theorem of Riemannian geometry.

Theorem 2.30. Let (M, g) be a Riemannian manifold. Then there exists a unique connection ∇ on M which for all $X, Y, Z \in \Gamma(\mathrm{T}M)$ satisfies

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

where $g(Y, Z)$ denotes the function $M \to \mathbb{R}$, $p \mapsto g_p(Y(p), Z(p))$.

Proof. First, we show uniqueness which suggests a formula to prove existence. Consider the three equations that arise from the assumption through cyclic permutation of the variables:

$$\begin{cases} Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \\ Yg(Z, X) = g(\nabla_Y Z, X) + g(Z, \nabla_Y X) \\ Zg(X, Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y). \end{cases}$$

Computing the sum of the first two right hand sides and subtracting the third yields

$$\begin{aligned} g(\nabla_X Y, Z) &+ g(\nabla_Y X, Z) + g(\nabla_X Z, Y) - g(\nabla_Z X, Y) + g(\nabla_Y Z, X) - g(\nabla_Z Y, X) \\ &= 2g(\nabla_X Y, Z) - g([X, Y], Z) + g([X, Z], Y) + g([Y, Z], X) \end{aligned}$$

Therefore, we may record that $2g(\nabla_X Y, Z)$ equals

$$Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) + g([X, Y], Z) - g([X, Z], Y) - g([Y, Z], X)$$

which shows uniqueness since g is non-degenerate. As for existence, let $T(X, Y, Z)$ denote the right hand side of the above equation which we write as

$$\underbrace{(g([X, Y], Z) - Zg(X, Y))}_{A(X,Y,Z)} + \underbrace{(Yg(Z, X) - g([Y, Z], X))}_{B(X,Y,Z)} + \underbrace{(Xg(Z, Y) - g([X, Z], Y))}_{B(Y,X,Z)}$$

Observe that $2g(\nabla_X Y, Z) = 2g(\nabla_X Y, Z)$. Now, it remains to verify that $T(X, Y, Z)(p)$ only depends on $Z(p)$ in which case the expression $T(X, Y, Z)(p)$ is a linear form in $Z(p)$ on T_pM for fixed X and Y, to that there exists a unique vector $\nabla_X Y(p) \in \mathrm{T}_pM$ such that $2g_p(\nabla_X Y(p), Z(p)) = T(X, Y, Z)(p)$.

In order to verify this, we check what happens if we replace Z by fZ for some $f \in C^\infty(M)$. Compute

$$\begin{aligned} A(X, Y, fZ) &= g([X, Y], fZ) + fZg(X, Y) \\ &= f(g([X, Y], Z) + Zg(X, Y)) \\ &= fA(X, Y, Z) \end{aligned}$$

as well as

$$\begin{aligned} B(X, Y, fZ) &= Yg(fZ, X) - g([Y, fZ], X) \\ &= Y(fg(Z, X)) - g(Y(f)Z + f[Y, Z], X) \\ &= Y(f)g(Z, X) + fYg(Z, X) - Y(f)g(Z, X) - fg([Y, Z], X) \\ &= f(Y_g(Z, X) - g([Y, Z], X)) \\ &= fB(X, Y, Z). \end{aligned}$$

This shows $T(X,Y,fZ) = fT(X,Y,Z)$ and hence as in Remark 2.29 implies that $T(X,Y,Z)(p)$ only depends on $Z(p)$ for given X and Y. As explained above we may hence define $\nabla_X Y(p)$ using the equation that showed uniqueness. It is left as an exercise to verify that the map ∇ so defined is indeed a connection. □

Definition 2.31. Let (M,g) be a Riemannian manifold. The connection of Theorem 2.30 is the *Levi-Civita connection* of (M,g).

We now aim to get a hold on the Levi-Civita connection by establishing a formula in local coordinates. This dates back to Einstein and his work with Grossmann at ETH Zurich around 1910. Recall that given a chart (U,φ) of M we have for every $p \in M$ a basis $(\partial_1(p),\dots,\partial_m(p))$ of T_pM given by

$$\partial_i(p)(f) = \frac{\partial(f\circ\varphi^{-1})}{\partial x_i}(\varphi(p)) \quad \text{for all } f \in C^\infty(M).$$

Now consider the vector fields $\partial_i \in \Gamma(\mathrm{T}U)$ and let ∇ denote the Levi-Civita connection of M. Then

$$\nabla_{\partial_j}\partial_k(p) = \sum_{l=1}^m \Gamma^l_{jk}\partial_l(p)$$

where the smooth functions $\Gamma^l_{jk} : U \to \mathbb{R}$ are called *Christoffel symbols* and which we now determine: Let $g = \sum_{i,j=1}^m g_{ij}dx_i \otimes dx_j$ be the expression of the Riemannian metric in the chart (U,φ). Recall in particular that $g_{ij}(p) = g_p(\partial_i(p),\partial_j(p))$. We are going to use the formula established in the proof of Theorem 2.30. To this end, observe that $[\partial_i,\partial_j] = 0$ on U because of the definition of the ∂_i and the fact that

$$\frac{\partial^2 g}{\partial x_i \partial x_j} = \frac{\partial^2 g}{\partial x_j \partial x_i} \quad \text{for all } g \in C^\infty(\varphi(U)).$$

Employing said formula we obtain

$$\begin{aligned} 2g(\nabla_{\partial_j}\partial_k,\partial_i) &= \partial_j g(\partial_k,\partial_i) + \partial_k g(\partial_i,\partial_j) - \partial_i g(\partial_j,\partial_k) \\ &= \partial_j g_{ki} + \partial_k g_{ij} - \partial_i g_{jk} \end{aligned}$$

Substituting the definition of the Christoffel symbols yields

$$2\sum_{l=1}^m \Gamma^l_{jk} g_{li} = \partial_j g_{ki} + \partial_k g_{ij} - \partial_i g_{jk}.$$

Let g^{-1} denote the matrix inverse of (g_{ij}). Multiplying the equation by $(g^{-1})_{ir}$ and summing over $i \in \{1,\dots,m\}$ we obtain using $\sum_{i=1}^m g_{li}(g^{-1})_{ir} = \delta_{lr}$:

$$2\Gamma^r_{jk} = \sum_{i=1}^m (g^{-1})_{ir}\left(\partial_j g_{ki} + \partial_k g_{ij} - \partial_i g_{jk}\right).$$

Proposition 2.32. Retain the above notation. Then

$$\nabla_X Y = \sum_{i=1}^m \left(\sum_j X_j \partial_j Y_i + \sum_{j,k} \Gamma^i_{jk} X_j Y_k \right) \partial_i$$

where

$$2\Gamma^l_{jk} = \sum_{i=1}^m (g^{-1})_{ir}\left(\partial_j g_{ki} + \partial_k g_{ij} - \partial_i g_{jk}\right).$$

Proof. We compute

$$\begin{aligned}
\nabla_X Y &= \nabla_{\sum_i X_i \partial_i}\left(\sum_k Y_k \partial_k\right) \\
&= \sum_{j,k} X_j \nabla_{\partial_j}(Y_k \partial_k) \\
&= \sum_{j,k} X_j \partial_j Y_k \cdot \partial_k + \sum_{j,k} X_j Y_k \nabla_{\partial_j} \partial_k \\
&= \sum_i \left(\sum_j X_j \partial_j Y_i\right) \partial_i + \sum_{j,k} X_j Y_k \sum_i \Gamma^i_{jk} \partial_i \\
&= \sum_i \left(\sum_j X_j \partial_j Y_i + \sum_{j,k} \Gamma^i_{jk} X_j Y_k\right) \partial_i.
\end{aligned}$$

□

Example 2.33. Consider the following two examples.

(i) Consider $(\mathbb{R}^n, \mathrm{can})$. Then $g^{(p)}_{ij} = \delta_{ij}$ for all $p \in \mathbb{R}^n$ whence $\Gamma^i_{jk} = 0$ for all $i, j, k \in \{1, \ldots, n\}$. We therefore have

$$\nabla_X Y = \sum_i \left(\sum_j X_j \partial_j Y_i\right) \partial_i.$$

In fact, $\nabla_{\partial_j} \partial_k = 0$ for all $j, k \in \{1, \ldots, n\}$.

(ii) Consider the upper-half-plane $H^2 = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ with the Riemannian metric $((dx)^2 + (dy)^2)/y^2$. For the computations, it is useful to think of x and y in terms of x_1 and x_2, and accordingly replace ∂_x and ∂_y with ∂_1 and ∂_2 respectively. We then get

$$2g(\nabla_{\partial_j} \partial_k, \partial_i) = \partial_j(\delta_{ki} x_2^{-2}) + \partial_k(\delta_{ij} x_2^{-2}) - \partial_i(\delta_{jk} x_2^{-2}).$$

For instance, let us compute $\nabla_{\partial_1} \partial_1$: We have $2g(\nabla_{\partial_1} \partial_1, \partial_1) = 0$ and

$$2g(\nabla_{\partial_1} \partial_1, \partial_2) = 2x_2^{-3} = 2x_2^{-1} g(\partial_2, \partial_2),$$

and therefore obtain

$$\nabla_{\partial_1} \partial_1 = x_2^{-1} \partial_2$$

and similarly

$$\nabla_{\partial_1} \partial_2 = \nabla_{\partial_2} \partial_1 = -x_2^{-1} \partial_1 \quad \text{and} \quad \nabla_{\partial_2} \partial_2 = -x_2^{-1} \partial_2.$$

Canonical Connection on a Riemannian Submanifold. A nice consequence of the formula for the Levi-Civita connection is that it is simple to compute the Levi-Civita connection of a Riemannian submanifold: Let (M, g) be a Riemannian submanifold of a Riemannian manifold (N, h), that is $M \subseteq N$ is embedded and for all $p \in M$ we have $g_p = h_p|_{\mathrm{T}_p M \times \mathrm{T}_p M}$.

Proposition 2.34. Retain the above notation and let ∇^M and ∇^N denote the Levi-Civita connections of (M, g) and (N, h) respectively. Further, let $X, Y \in \Gamma(\mathrm{T}M)$ and $X', Y' \in \Gamma(\mathrm{T}N)$ be smooth vector fields with $X'|_M = X$ and $Y'|_M = Y$. Then

$$(\nabla^M_X Y)(p) = \left(\nabla^N_{X'} Y'(p)\right)^{\perp_p}$$

for all $p \in M$ where $\perp_p$ denotes the orthogonal projection from $\mathrm{T}_p N$ to $\mathrm{T}_p M$.

Proof. Let (U, φ) be a chart of N at $p \in M \subseteq N$ such that $\varphi(U) = C^n_\varepsilon(0)$ and $\varphi(U \cap M) = C^m_\varepsilon(0) \times (0)^{n-m}$. Now consider $X, Y, Z \in \Gamma(\mathrm{T}M)$ and let X', Y' and Z' be extensions of X, Y and Z to open neighbourhoods of M in N. Then $X' = \sum_{i=1}^n X'_i \partial_i$ with $X'_i|_{U\cap M} = 0$ for all $i \in \{m+1, \ldots, n\}$ and similarly for Y' and Z'. Now recall the formular for the bracket of vector fields in local coordinates:

$$[X', Y'] = \sum_{i,j} \left(X'_i \partial_i Y'_j - Y'_i \partial_i X'_j \right) \partial_j$$

which, taking into account the vanishing of X'_i and Y'_i on $U \cap M$ for $i \geq m+1$, implies: $[X', Y']_{U\cap M} = [X, Y]$. The formula in the proof of Theorem 2.30 now implies for $p \in U \cap M$:

$$2h_p(\nabla^N_{X'} Y'(p), Z'(p)) = 2g_p(\nabla^M_X Y(p), Z(p))$$

which implies the proposition since $Z'(p) = Z(p)$. □

Covariant Derivative Along a Curve. In this section we discuss the covariant derivative along curves which ultimately leads to the important notion of geodesics.

Definition 2.35. Let $c : I \to M$ be a smooth curve defined on some open interval $I \subseteq \mathbb{R}$. A *vector field along* c is a smooth map $X : I \to \mathrm{T}M$ with $X(t) \in \mathrm{T}_{c(t)}M$.

Retaining the above notation, our aim is to define "$\nabla_{c'(t)}X$" for a vector field X along c. Let $\Gamma(c^*\mathrm{T}M)$ denote the vector space of such vector fields. It is also a $C^\infty(I)$-module.

Remark 2.36. The notation $c^*\mathrm{T}M$ stands for the pullback of the tangent bundle $\mathrm{T}M$ by c. It is defined as

$$c^*\mathrm{T}M = \{(t, v) \in I \times \mathrm{T}M \mid v \in \mathrm{T}_{c(t)}M\}.$$

Then $c^*\mathrm{T}M$ is a smooth vector bundle with base I and a smooth vector field along c is merely a smooth section of $c^*\mathrm{T}M$.

Theorem 2.37. Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇ and let $c : I \to M$ be a smooth curve defined on an open interval $I \subseteq \mathbb{R}$. Then there is a linear map

$$\frac{D}{dt} : \Gamma(c^*\mathrm{T}M) \to \Gamma(c^*\mathrm{T}M)$$

such that for all $f \in C^\infty(I)$ and $Y \in \Gamma(c^*\mathrm{T}M)$:

$$\frac{D}{dt}(fY) = f'Y + f\frac{DY}{dt}$$

If furthermore $X \in \Gamma(\mathrm{T}M)$ satisfies $Y(t) = X(c(t))$ for all $t \in I$ we have

$$\frac{DY}{dt}(t) = (\nabla_{c'(t)}X)(c(t)).$$

Proof. Let $Y \in \Gamma(c^*(\mathrm{T}M))$ be such that $Y(t) = X(c(t))$ for some $X \in \Gamma(\mathrm{T}M)$. We compute $(\nabla_{c'(t)}X)(c(t))$ in a chart (U, φ): Let $\varphi(c(t)) = (x_1(t), \ldots, x_m(t))$. Then

$$c'(t) = (D_t c)(1) = \sum_{i=1}^m x'_i(t) \partial_i(c(t)).$$

Indeed, we have

$$\begin{aligned} c'(t)(f) &= (D_{c(t)} f)(c'(t)) \\ &= D_{c(t)}((f \circ \varphi^{-1}) \circ \varphi)(c'(t)) \\ &= D_{\varphi(c(t))}(f \circ \varphi^{-1})(D_{c(t)}\varphi)(c'(t)) \\ &= D_{x(t)}(f \circ \varphi^{-1}) D_t(\varphi \circ c)(1) \\ &= \sum_{i=1}^{m} \frac{\partial (f \circ \varphi^{-1})}{\partial x_i}(x(t)) x_i'(t) \\ &= \sum_{i=1}^{m} x_i'(t) \partial_i(c(t))(f). \end{aligned}$$

Using Proposition 2.32 we therefore have

$$(\nabla_{c'(t)} X)(c(t)) = \sum_{i=1}^{m} \left(\sum_j x_j'(t) \partial_j X_i(c(t)) + \sum_{j,k} \Gamma^i_{jk}(c(t)) x_j'(t) X_k(c(t)) \right) \partial_i.$$

Now observe that $\sum_j x_j'(t)\partial_j X_i(c(t)) = Y_i'(t)$ whence

$$(\nabla_{c'(t)} X)(c(t)) = \sum_i \left(Y_i'(t) + \sum_{j,k} \Gamma^i_{jk}(c(t)) x_j'(t) Y_k(t) \right) \partial_i$$

Given the above computation, we now define for any $Y \in \Gamma(c^*\mathrm{T}M)$ and local coordinates (U, φ):

$$\frac{DY}{dt}(t) := \sum_{i=1}^{m} \left(Y_i'(t) + \sum_{j,k} \Gamma^i_{jk}(c(t)) x_j'(t) Y_k(t) \right) \partial_i$$

and leave the verifications to the reader. □

Theorem 2.37 allows us to make the following, central definition.

Definition 2.38. Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇ and let $c : I \to M$ be a smooth curve defined on an open interval $I \subseteq \mathbb{R}$. Further, let $D/dt : c^*(\mathrm{T}M) \to c^*(\mathrm{T}M)$ be the associated linear map and $X \in \Gamma(c^*(\mathrm{T}M))$. Then X is *parallel* if $D/dt(X) = 0$.

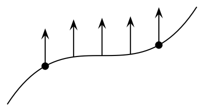
Example 2.39. Consider the Riemannian manifold $(\mathbb{R}^n, \mathrm{can})$. Let $c : \mathbb{R} \to \mathbb{R}^n$ be smooth and write $c(t) = (x_1(t), \dots, x_n(t))$. Then the map $\mathbb{R} \to \mathrm{T}\mathbb{R}^n$ given by $t \mapsto c'(t)$ is a smooth vector field along c. We have

$$\frac{Dc'}{dt}(t) = \sum_{i=1}^{n} x_i''(t) \partial_i(c(t)) = c''(t).$$

Thus, Dc'/dt is the acceleration.

If now $M \subseteq \mathbb{R}^n$ is a regular submanifold equipped with the induced Riemannian metric g and if $c : \mathbb{R} \to M$ is a smooth curve then $Dc'/dt = (c''(t))^\perp$; that is, the covariant derivative of $c'(t)$ is the orthogonal projection of $c''(t)$ onto $\mathrm{T}_{c(t)}M$.

We now turn to the notion of parallel transport, which in Euclidean space looks as follows.



Proposition 2.40. Let $c : I \to M$ be a C^1-curve defined on an open interval $I \subseteq \mathbb{R}$, and let $t_0 \in I$. Then for every $v \in \mathrm{T}_{c(t_0)}M$ there exists a unique parallel vector field $X : I \to \mathrm{T}M$ along c such that $X(t_0) = 0$.

Proof. It suffices to prove the Proposition for every compact subinterval $[t_0, T] \subseteq \mathbb{R}$. Let $t_0 < t_1 < \cdots < t_n = T$ be such that $c([t_i, t_{i+1}]) \subseteq U_i$ where (U_i, φ_i) is a chart. We now look at the condition for X to be parallel in the chart (U, φ): The condition $DX/dt = 0$ is equivalent to the system

$$X_i'(t) + \sum_{j,k} \Gamma^i_{jk}(c(t))x_j'(t)X_k(t) = 0 \qquad (i \in \{1, \ldots, m\})$$

for all t in an open interval $J \subseteq \mathbb{R}$ with $c(J) \subseteq U$. For $t' \in J$ and $(v_1, \ldots, v_m) \in \mathbb{R}^n$ there is a unique solution of the above system on the whole of J with $X_k(t') = v_k$. Applying this argument to each subinterval $[t_i, t_{i+1}]$ yields the conclusion. □

Definition 2.41. Let $c : [a, b] \to M$ be a C^1-map. The *parallel transport* $P_{c,a,b}$ is the linear map $\mathrm{T}_{c(a)}M \to \mathrm{T}_{c(b)}M$ which to every $v \in \mathrm{T}_{c(a)}M$ associates $X(b) \in \mathrm{T}_{c(b)}$ where $X : I \to \mathrm{T}M$ is the parallel vector field with $X(a) = v$ and c is the restriction of a C^1-map defined on an open interval $I \subseteq \mathbb{R}$ containing $[a, b]$.

Proposition 2.42. Retain the notation of Definition 2.41. Then the parallel transport $P_{c,a,b} : \mathrm{T}_{c(a)}M \to \mathrm{T}_{c(b)}M$ is an isometry. More generally, if $X, Y \in \Gamma(c^*\mathrm{T}M)$ then

$$\frac{d}{dt} g_{c(t)}(X(t), Y(t)) = g_{c(t)}\left(\frac{D}{dt}X(t), Y(t)\right) + g_{c(t)}\left(X(t), \frac{D}{dt}Y(t)\right).$$

Proof. The general formula is left as an exercise. To deduce that parallel transport is an isometry, note that if X and Y are parallel we obtain $(d/dt)g(X(t), Y(t)) = 0$ and hence $g(X(a), Y(a)) = g(X(b), Y(b))$. □

One can recover the Levi-Civita connection from the parallel transport.

Proposition 2.43. Let (M, g) be a Riemannian manifold, $X, Y \in \Gamma(\mathrm{T}M)$, $p \in M$ and $c : (-\varepsilon, \varepsilon) \to M$ an integral curve of X with $c(0) = p$. Then

$$(\nabla_X Y)(p) = \left.\frac{d}{dt}\right|_{t=0} P^{-1}_{c,0,t}(Y(c(t)))$$

Proof. (Sketch). Define an operator $D : \Gamma(\mathrm{T}M) \times \Gamma(\mathrm{T}M) \to \Gamma(\mathrm{T}M)$ by

$$D_X Y(p) = \left.\frac{d}{dt}\right|_{t=0} P^{-1}_{c,0,t}(Y(c(t)))$$

and argue that $D = \nabla$ using the uniqueness of the Levi-Civita connection: Consider another $Z \in \Gamma(\mathrm{T}M)$. Since $P_{c,0,t}$ is an isometry we have

$$g(P^{-1}_{c,0,t}(Y(c(t))), P^{-1}_{c,0,t}(Z(c(t)))) = g(Y(c(t)), Z(c(t))).$$

Taking the derivative at $t = 0$ yields:

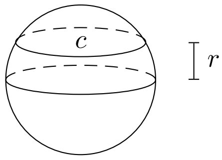
$$Xg(Y, Z) = g(D_X Y, Z) + g(Y, D_X Z).$$

at p. Also, one verifies that $D_{fX}Y = fD_X Y$ and $D_X fY = X(f)Y + fD_X Y$. □

Example 2.44. Consider the sphere $S^2 = \{x \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}$ with its induced Riemannian metric. In this example, we compute the parallel transport along the circle

$$c : [0, 2\pi] \to S^2,\ t \mapsto (\sqrt{1 - r^2}\cos t, \sqrt{1 - r^2}\sin t, r)$$

for $r \in [0, 1)$. Specifically, we determine the map $T_{c,0,2\pi} : \mathrm{T}_{c(0)}S^2 \to \mathrm{T}_{c(0)}S^2$.



To this end, let $x(t) = (x_1(t), x_2(t), x_3(t)) \in \mathrm{T}_{c(t)}S^2$. That is, for all $t \in [0, 2\pi]$:

$$x_1(t)\sqrt{1 - r^2}\cos t + x_2(t)\sqrt{1 - r^2}\sin t + x_3(t)r = 0. \tag{1}$$

According to Example 2.39, the parallelity condition $Dx/dt = 0$ translates to $x'(t)$ being orthogonal to $\mathrm{T}_{c(t)}S^2$ in this context. That is

$$\frac{x_1'(t)}{\sqrt{1 - r^2}\cos t} = \frac{x_2'(t)}{\sqrt{1 - r^2}\sin t} = \frac{x_3'(t)}{r} \tag{2}$$

Taking the derivative of (1) with respect to t and substituting expressions for the $x_1'(t)$ and $x_2'(t)$ summands in terms of x_3' resulting from (2) yields

$$-x_1(t)\sqrt{1 - r^2}\sin t + x_2(t)\sqrt{1 - r^2}\cos t + \frac{x_3'(t)}{r} = 0. \tag{3}$$

Taking the derivative of (3) and taking into account (1) as well as (2) yields

$$x_3'' + rx_3 = 0$$

Hence $x_3(t) = A\cos(rt) + B\sin(rt)$. From this one obtains using (2) formulas for x_1' and x_2' which can be integrated, leading to

$$\begin{aligned} x_1(t) =& \frac{-A}{\sqrt{1 + r^2}}(\sin(t)\sin(rt) + r\cos(t)\cos(rt)) \\ &+ \frac{B}{\sqrt{1 - r^2}}(\sin(t)\cos(rt) - r\cos(t)\sin(rt)) + c_1 \end{aligned}$$

and

$$\begin{aligned} x_2(t) =& \frac{-A}{\sqrt{1 + r^2}}(-\cos(t)\sin(rt) + r\sin(t)\cos(rt)) \\ &- \frac{B}{\sqrt{1 - r^2}}(\cos(t)\cos(rt) + r\sin(t)\sin(rt)) + c_2. \end{aligned}$$

The orthogonality condition $\langle x(t), c(t)\rangle = 0$ leads to $c_1 = 0 = c_2$. Finally, we get

$$x(0) = \left(\frac{-Ar}{\sqrt{1 - r^2}}, \frac{-Br}{\sqrt{1 - r}}, A\right),$$

$$\begin{aligned} x(2\pi) = \Bigg(& \frac{-r}{\sqrt{1 - r^2}}\left(A\cos(2\pi r) + B\sin(2\pi r)\right), \\ & \frac{r}{\sqrt{1 - r^2}}\left(A\sin(2\pi r) - B\cos(2\pi r)\right), \\ & A\cos(2\pi r) + B\sin(2\pi r)\Big), \end{aligned}$$

Thus, if we take $v_1 := (r, 0, -\sqrt{1-r^2})$ and $v_2 := (0, 1, 0)$ as a basis of $\mathrm{T}_{c(0)}S^2$, the matrix of $T_{c,0,2\pi}$ is given by

$$\begin{pmatrix} \cos(2\pi r) & \sin(2\pi r) \\ -\sin(2\pi r) & \cos(2\pi r) \end{pmatrix}.$$

2.3. **Geodesics.** In this section, we finally discuss the important notion of geodesics. Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇. Furthermore, let D/dt denote the covariant derivative along a smooth curve c.

Definition 2.45. Retain the above notation. Let $c : I \to M$, defined on an open interval $I \subseteq \mathbb{R}$, be a smooth curve. Then c is a *geodesic* if $Dc'/dt = 0$ where $c' : I \to \mathrm{T}M$ is the tangent velocity vector field along c.

In other words, a geodesic is a curve whose tangent vector field is invariant with respect to parallel transport along itself. Intuitively, the position vector does not experience any acceleration. We will shed light on the exact relation between geodesics and distance-minimising curves later on but for the moment, convince yourself that just as in driving, accelerating corresponds to making a detour.

Geodesics can also be looked at from a variational point of view: Given points $x, y \in M$, consider all curves c connecting x to y and the functional L which to a given curve associates its length. If one sets up things correctly then the Euler-Lagrange equations arise from the condition that a curve c be a critical point of the functional L, see e.g. [Spi79].

In local coordinates (U, φ) with $\varphi(c(t)) = (x_1(t), \ldots, x_m(t))$, the functions x_i ($i \in \{1, \ldots, m\}$) associated to the geodesic c are the solution of the differential equation

$$x_i''(t) + \sum_{j,k} \Gamma^i_{jk}(x(t))x_k'(t)x_j'(t) = 0. \tag{GLC}$$

Example 2.46. Before going into the theory of geodesics, we collect some examples.

(i) Let $M \subseteq (\mathbb{R}^n, \mathrm{can})$ be a regular submanifold with the induced Riemannian metric. Then a smooth curve $c : I \to M \subseteq \mathbb{R}^n$, defined on an open interval $I \subseteq \mathbb{R}$ is a geodesic in M if and only if the acceleration vector $c''(t)$ is orthogonal to $\mathrm{T}_{c(t)}M$ for all $t \in I$: Indeed, if ∇ is the Levi-Civita connection on M and $\overline{\nabla}$ is the one on $\mathbb{R}^n$ we know that $c''(t) = \overline{D}c'/dt$ and $Dc'/dt = (\overline{D}c'/dx)^{\perp_{c(x)}}$.

(ii) Geodesics in $(\mathbb{R}^n, \mathrm{can})$ are of the form $c(t) = tv + w$ for some $v, w \in \mathbb{R}^n$ as in this case (GLC) simply says $x_i''(t) = 0$.

(iii) Consider $S^n \subseteq \mathbb{R}^{n+1}$ with the induced Riemannian metric. A great circle is determined by a pair $u, v \in S^n$ of orthogonal vectors via

$$c(\theta) = \cos(\theta)u + \sin(\theta)v.$$

Then $c'(\theta) = -\sin(\theta)u + \cos(\theta)v$ and $c''(\theta) = -\cos(\theta)u - \sin(\theta)v = -c(\theta)$. As a result, $\overline{D}c'/dt$ is orthgonal to $\mathrm{T}_{c(\theta)}S^n$ and hence c is a geodesic of S^n.

The local existence and uniqueness of geodesics follows from the following well-known theorem.

Theorem 2.47. Let $\Omega \subseteq \mathbb{R}^n$ be open and let $F : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ be a smooth map. Then for every $(s_0, v_0) \in \Omega \times \mathbb{R}^n$ there is a neighbourhood $E \times V \subseteq \Omega \times \mathbb{R}^n$ of (s_0, v_0) and $\delta > 0$ such that for every $(s, v) \in E \times V$ there is a unique smooth curve $x_{s,v} : (-\delta, \delta) \to \Omega$ satisfying

(i) $x''(t) = F(x(t), x'(t))$,
(ii) $x(0) = s$ and $x'(0) = v$.

Furthermore, the map $E \times V \times (-\delta, \delta) \to \Omega$, $(s, v, t) \mapsto x_{s,v}(t)$ is smooth.

We now apply Theorem 2.47 to our situation.

Corollary 2.48. Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇. Then every $p_0 \in M$ admits a neighbourhood $U \subseteq M$, $\delta > 0$ and $\varepsilon > 0$ such that for every $p \in U$ and $v \in \mathrm{T}_p M$ with $|v| < \varepsilon$ there is a unique geodesic $c_{(p,v)} : (-\delta, \delta) \to M$ such that $c_{(p,v)}(0) = p$ and $c'_{(p,v)}(0) = v$. Moreover, the map

$$C : \{(p, v) \in \mathrm{T}U \mid |v| < \varepsilon\} \times (-\delta, \delta) \to M,\ ((p, v), t) \to c_{(p,v)}(t)$$

is smooth.

The following homogeneity lemma often occurs underrated in the literature. It critically depends on the form of (GLC).

Lemma 2.49. Retain the notation of Corollary 2.48. Assume that the geodesic $c_{(p,v)}$ is defined on $(-\delta, \delta)$ and let $a > 0$. Then $c_{(p,av)}$ is defined on $(-\delta/a, \delta/a)$ and $c_{(p,av)}(t) = c_{(p,v)}(at)$.

Proof. To simplify notation, we put $\eta(t) = c_{(p,v)}(at)$ and $\gamma(t) = c_{(p,v)}(t)$. Observe that $\eta(0) = p$ and $\eta'(0) = av$. We show that η is a geodesic, i.e. $D\eta'/dt = 0$ where the covariant derivative is taken along η. Taking a local chart (U, φ) and setting

$$\varphi(\gamma(t)) + (x_1(t), \dots, x_m(t)) \quad \text{and} \quad \varphi(\eta(t)) = (y_1(t), \dots, y_m(t))$$

We know that $y_i(t) = x_i(at)$. Therefore:

$$\begin{aligned} y_i''(t) = \sum_{j,k} \Gamma^i_{jk}(\eta(t)) y_k'(t) y_j'(t) &= a^2 x_i''(at) + \sum_{j,k} \Gamma^i_{jk}(\gamma(at)) a x_k'(at) a x_j'(at) \\ &= a^2 \left(x_i''(at) + \sum_{j,k} \Gamma^i_{jk}(\gamma(at)) x_k'(at) x_j'(at) \right) = 0 \end{aligned}$$

which proves the assertion. □

Corollary 2.50. Retain the above notation. Then every $p_0 \in M$ admits a neighbourhood $U \subseteq M$ and $\varepsilon > 0$ such that for all $p \in U$ and $v \in \mathrm{T}_p U$ with $|v| < \varepsilon$ there is a unique geodesic $c_{(p,v)} : (-2, 2) \to M$ with $c_{(p,v)}(0) = p$ and $c'_{(p,v)}(0) = v$.

Proof. This follows from the homogeneity Lemma 2.49: If $c_{(p,v)}$ is defined on $(-\delta, \delta)$ then $\varepsilon_{(p,\delta v/2)}$ is defined on $(-2, 2)$. □

In the context of Corollary 2.50, let

$$\Omega := \{(p, v) \in \mathrm{T}M \mid c_{(p,v)} \text{ is defined on } (-2, 2)\}.$$

We have seen that Ω is a neighbourhood of the trivial section of $\mathrm{T}M$.

Definition 2.51. Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇ and retain the above notation. The *exponential map* is

$$\exp : \Omega \to M,\ \exp((p, v)) = c_{(p,v)}(1).$$

We denote its restriction to $\Omega \cap \mathrm{T}_p M$ by $\exp_p$.

As a consequence of Corollary 2.50, the exponential map is smooth.

Proposition 2.52. Retain the above and set $\phi : \Omega \to M \times M$, $(p, v) \mapsto (p, \exp_p v)$. Then ϕ is smooth and for every $(p_0, 0) \in \Omega$ the map ϕ is a diffeomorphism from a neighbourhood of $(p_0, 0)$ to its image in $M \times M$.

Proof. The map ϕ is smooth because its components are. For the second statement we employ the inverse function theorem for which end we have to compute $D_{(p_0,0)}\phi$: Let (U, φ) be a chart of M at $p_0 \in M$ and recall that it yields a basis $(\partial_1(p), \dots, \partial_m(p))$ of T_pU for all $p \in U$. We will also use the diffeomorphism

$$U \times \mathbb{R}^m \to \mathrm{T}U,\ (p, y) \mapsto \left(p, \sum_{i=1}^{m} y_i \partial_i(p)\right)$$

Slightly pedantic, we put $\psi(p, y) := \phi(p, v)$ where $v = \sum_{i=1}^m y_i\partial_i(p)$. Finally, we identify $\mathrm{T}_{(p,0)}(U \times \mathbb{R}^m)$ with $\mathrm{T}_pU \oplus \mathbb{R}^m$ where we take

$$(\partial_1(p), 0), \dots, (\partial_m(p), 0), (0, \partial_1 y), \dots, (0, \partial_m y)$$

with $\partial_i y = \partial/\partial y_i|_{y=0}$ as basis of the right hand space. Now, let's consider the variation of ψ in the direction of U, keeping the other fixed: We have

$$\psi(c_{(p,v)}(t), 0) = (c_{(p,v)}(t), c_{(p,v)}(t))$$

and therefore $D_{(p,0)}\psi(v, 0) = (v, v) \in \mathrm{T}_pM \oplus \mathrm{T}_pM$. For the other variation, consider the smooth path $t \mapsto (p, ty)$ in $U \times \mathbb{R}^m$ with tangent vector $(0, y) \in \mathrm{T}_pM \oplus \mathbb{R}^m$. Then

$$\psi(p, ty) = (p, c_{(p,tv)}(1)) = (p, c_{(p,v)}(t))$$

where $v = \sum_{i=1}^m y_i\partial_i(p)$. Thus $D_{(p,0)}\psi(0, y) = (0, v)$. Identifying $\mathrm{T}_pM \times \mathrm{T}_pM$ with $\mathbb{R}^m \oplus \mathbb{R}^m$ using the basis

$$((\partial_1(p), 0), \dots, (\partial_m(p), 0), (0, \partial_1(p)), \dots, (0, \partial_m(p)))$$

the matrix of $D_{(p,0)}\psi$ is

$$D_{(p,0)}\psi = \begin{pmatrix} \mathrm{Id} & 0 \\ \mathrm{Id} & \mathrm{Id} \end{pmatrix}.$$

□

Corollary 2.53. Retain the above notation. For every $p_0 \in M$ there is an open neighbourhood U of p_0 in M and $\varepsilon > 0$ such that the following hold.

(i) For every $(x, y) \in U \times U$ there is a unique $v \in \mathrm{T}_xM$ with $|v| < \varepsilon$ and $\exp_x(v) = y$.

In this case, let $c(x, y, t) := \exp_x(tv)$.

(ii) The map $U \times U \times (-2, 2) \to M,\ (x, y, t) \mapsto c(x, y, t)$ is defined and smooth.
(iii) For every $x \in U$ the map $\exp_x : B(0, \varepsilon) \to M$ is a diffeomorphism onto its image.

Proof. Retaining the above notation, let $W \subseteq \Omega$ be an open neighbourhood of $(p_0, 0) \in \Omega$ such that $\phi : W \to \phi(W)$ is a diffeomorphism. It is easy to see from local triviality of $\mathrm{T}M$ that there exists an open neighbourhood V of $p_0 \in M$ and $\varepsilon > 0$ such that $\mathrm{T}_{<\varepsilon}V := \{(p, v) \in \mathrm{T}M \mid p \in V,\ |v| < \varepsilon\} \subseteq W$. In particular, $\phi(\mathrm{T}_{<\varepsilon}V)$ is an open neighbourhood of $(p_0, p_0) = \phi(p_0, 0) \subseteq M \times M$. Hence there is an open neighbourhood U of $p_0 \in M$ with $U \times U \subseteq \phi(\mathrm{T}_{<\varepsilon}V)$. In particular $U \subseteq V$. This pair (U, ε) satisfies the conclusions. □

The infinitesimal definition of geodesics has the disadvantage that its relation to distance-minimising curves between points remains unclear at the moment. On the plus side, it enables one to prove the following which certainly would not hold if geodesics had been defined as curves of shortest lengths between points: Think of the projection of a straight line onto the torus.

Corollary 2.54. Let (N, h) and (M, g) be Riemannian manifolds. Furthermore, let $p : (N, h) \to (M, g)$ be a Riemannian covering. Given a smooth curve $c : I \to M$, let $\widetilde{c} : I \to N$ be any lift of c, i.e. $p \circ \widetilde{c} = c$. Then c is a geodesic if and only if $\widetilde{c}$ is.

For instance, Corollary 2.54 can be used to understand the geodesics of Bieberbach manifolds.

2.4. **Bi-invariant Metrics on Lie Groups.** In this section, we illustrate many of the notions introduced so far for Lie groups G that admit a bi-invariant metric. Recall that this means that all left- and right-translations of G are isometries. In turn, this implies that $\langle -,-\rangle_e$ is invariant under $\mathrm{Ad}(G)$. Conversely, recall that an $\mathrm{Ad}(G)$-invariant scalar product on T_eG gives rise to a bi-invariant metric via left- and right-translation.

Example 2.55. We recall that any compact Lie group admits a bi-invariant metric. Thus all orthogonal groups do. For instance, consider $G = \mathrm{O}(n) \subseteq \mathrm{GL}(n,\mathbb{R})$ with

$$\mathrm{Lie}(\mathrm{O}(n)) = \mathrm{T}_{\mathrm{Id}}G = \{X \in M_{n,n}(\mathbb{R}) \mid X + X^T = 0\}.$$

As for all matrix Lie groups, we have $\mathrm{Ad}(g)X = gXg^{-1}$. Given $X, Y \in \mathrm{T}_eG = \mathrm{Lie}(G)$, set $\langle X, Y\rangle = -\operatorname{tr}(XY)$. This is a bilinear, symmetric form. In addition, we have

$$\langle X, \rangle X = -\operatorname{tr}(XX) = \operatorname{tr}((-X)X) = \operatorname{tr}(X^TX) = \sum_{i,j} x_{ij}^2 \geq 0$$

which proves positive-definiteness. Furthermore,

$$\langle \mathrm{Ad}(g)X, \mathrm{Ad}(g)Y\rangle = -\operatorname{tr}(gXg^{-1}gYg^{-1}) = -\operatorname{tr}(gXYg^{-1}) = -\operatorname{tr}(XY) = \langle X, Y\rangle.$$

Hence $\langle -,-\rangle_e$ determines a bi-invariant metric on $\mathrm{O}(n)$.

We now show that for the class of Lie groups under consideration, the Riemannian and the Lie group exponential maps coincide which yields a very explicit understanding of all geodesics, particularly for matrix Lie groups.

Theorem 2.56. Let G be a Lie group admitting a bi-invariant Riemannian metric. Then the Riemannian exponential is defined on the whole of $\mathrm{T}G$ and its restriction $\exp_e : \mathrm{T}_eG \to G$ coincides with the Lie group exponential.

Our proof of Theorem 2.56 relies on the following two lemmas.

Lemma 2.57. Let G be a Lie group admitting a bi-invariant Riemannian metric. Then $i : G \to G$, $g \mapsto g^{-1}$ is an isometry.

Proof. We write the relation $gi(g) = e$ in the following way: The map $m \circ \Delta$ is constant and equal to $e \in G$ where $\Delta : G \to G \times G$, $g \mapsto (g, i(g))$. In particular, we have for all $h \in G$ and $v \in \mathrm{T}hG$: $D_h(m \circ \Delta)(v) = 0$. Note that $D_h(m \circ \Delta) = D_{\Delta(h)}m \circ D_h\Delta$ and

$$D_{(h_1,h_2)}m(v_1, v_2) = D_{h_1}R_{h_2}(v_1) + D_{h_2}L_{h_1}(v_2).$$

Indeed, consider for instance $m(c_{v_1}(t), h_2) = c_{v_1}(t)h_2$. Also, $D_h\Delta(v) = (v, \Delta_h i(v))$. Hence $0 = D_{(h,h^{-1})}m(v, D_hi(v)) = D_hR_{h^{-1}}(v) + D_{h^{-1}}L_h(D_hi(v))$ whence

$$D_{h^{-1}}L_h(D_hi(v)) = -D_hR_{h^{-1}}(v)$$

which proves the assertion. More neatly, we record $D_hi = -D_eL_{h^{-1}} \circ D_hR_{h^{-1}}$. In particular, $D_ei = -\mathrm{Id}$. □

Lemma 2.58. Retain the above notation. For every $v \in \mathrm{T}_eG$ the maximal interval of definition of the geodesic $c_{(e,v)}$ is the whole of $\mathbb{R}$, and $c_{(e,v)}(t_1 + t_2) = c_{(e,v)}(t_1)c_{(e,v)}(t_2)$ for all $t_1, t_2 \in \mathbb{R}$.

Proof. Fix $v \in \mathrm{T}_e G$ and let (a,b) containing $0 \in \mathbb{R}$ be the maximal interval of definition of $c_{(e,v)}$. Since i is an isometry with $D_e i = -\mathrm{Id}$ we have that $i(c_{(e,v)})$ is a geodesic with velocity $-v$ at $t=0$. Hence we have $i(c_{(e,v)}) = c_{(e,-v)}$ by uniqueness which implies that, whenever defined, $c_{(e,v)}(t)^{-1} = i(c_{(e,v)}(t)) = c_{(e,-v)}(t) = c_{(e,v)}(t)$. Thus $a=-b$. Now let $\varepsilon>0$ be such that $\exp_e : B(0,\varepsilon) \to G$ is a diffeomorphism onto its image and consider for $0<t_0<\varepsilon$: $\Gamma(s) = c(t_0)c(s) = L_{c(t_0)}c(s)$. By left-invariance of the metric, $\Gamma(s)$ is a geodesic which is defined on $(-b,b)$. Note that $\Gamma(-t_0) = c(t_0)c(-t_0) = e$ and $\Gamma(0) = c(t_0)$ and observe that $s \mapsto c(t_0+s)$ is a geodesic which at $s=-t_0$ takes the value $e \in G$ and at $s=0$ the value $c(t_0)$. By Corollary 2.53 we deduce that $\Gamma(s) = c(s+t_0)$. Since Γ is defined on $(-b,b)$, this would extend the domain of definition of c beyond b if b was finite since $t_0+b>b$. This implies $b=\infty$, i.e. $c_{(e,v)}$ is defined on the whole of $\mathbb{R}$ and

$$c_{(e,v)}(t_0)c_{(e,v)}(s) = c_{(e,v)}(t_0+s) \quad \text{for all } s \in \mathbb{R},\ |t_0|<\varepsilon.$$

Now let t be arbitrary and take $n \in \mathbb{N}\setminus\{0\}$ such that $|t|/n<\varepsilon$. Then

$$\begin{aligned} c_{(e,v)}(t+s) = c_{(e,v)}\left(\frac{t}{n} + \frac{(n-1)t}{n} + s\right) &= c_{(e,v)}\left(\frac{t}{n}\right) c_{(e,v)}\left(\frac{(n-1)t}{n} + s\right) \\ &= c_{(e,v)}\left(\frac{t}{n}\right)^n c(s). \end{aligned}$$

In particular, for $s=0$ we obtain $c_{(e,v)}(t) = c_{(e,v)}(t/n)^n$ which plugged into the above implies $c_{(e,v)}(t+s) = c(t)c(s)$. ☐

The proof of Theorem 2.56 is now no longer difficult.

Proof. (Theorem 2.56). Let $v \in \mathrm{T}_e G$ and let X_v be the left-invariant vector field determined by v. Furthermore, let $c_{(e,v)}$ be the geodesic determined by v. We need to show that $c_{(e,v)}$ is an integral curve for X_v: Write $c_{(e,v)}(s)c_{(e,v)}(t) = c(s+t)$, i.e. $L_{c_{(e,v)}(s)}(c_{(e,v)}(t)) = c_{(e,v)}(s+t)$. Taking the derivative with respect to t yields

$$D_{c_{(e,v)}} L_{c_{(e,v)}(s)}(c'_{(e,v)}(t)) = c'_{(e,v)}(s+t).$$

Evaluating at $t=0$ proves indeed $X_v(c(s)) = D_e L_{c_{(e,v)}(s)}(c'_{(e,v)}(0)) = c'(s)$. ☐

Example 2.59. Returning to our example $\mathrm{O}(n)$, the Riemannian exponential map is given by $\mathrm{Lie}(\mathrm{O}(n)) \to \mathrm{O}(n)$, $X \mapsto \sum_{n=0}^{\infty} X^n/n!$. In particular,

$$c_{(\mathrm{Id},X)}(t) = \sum_{n=0}^{\infty} \frac{t^n X^n}{n!}.$$

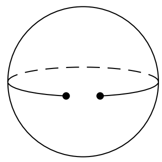
Therefore, $\mathrm{length}(c_{(\mathrm{Id},X)}[0,T]) = T\|X\|$. The matrix exponential can be computed via diagonalization and conjugation. In the present example, one can for instance use this to show that the exponential map is surjective and to determine periodic geodesics and their length.

Corollary 2.60. Let G be a Lie group admitting a bi-invariant Riemannian metric. Let ∇ denote its Levi-Civita connection and let X, Y be left-invariant vector fields on G. Then $\nabla_X Y = [X,Y]/2$.

Proof. Let Z be any left-invariant vector field and $c : \mathbb{R} \to G$ any integral curve of Z. If $c(0)=e$ then c is a geodesic. If $c(0) \neq e$ it is nevertheless a geodesic by left-invariance of Z, thus $Dc'/dt = 0$. Hence $\nabla_{c'(t)} Z = 0$ since $c'(t) = Z(c(t))$. For the same reason, $\nabla_Z Z = 0$. In particular $\nabla_{X+Y}(X+Y) = 0$ which by expanding implies $0 = \nabla_X Y + \nabla_Y X = 0$. Combining this with the fact that $\nabla_X Y - \nabla_Y X = [X,Y]$ yields the assertion. ☐

Later on, we will also examine the Riemannian curvature tensor in this setting and use it to derive global properties, e.g. information on the fundamental group.

2.5. **Geodesics and Distance.** In this section, we characterize geodesics as being locally length minimizing. The example of the sphere and the torus show that in general geodesics are not globally length-minimizing.



Now, let (M, g) be a Riemannian manifold and fix $p_0 \in M$. An open neighbourhood U of $p_0 \in M$ for which there is $\varepsilon > 0$ satisfying Corollary 2.53 is *totally normal*. Recall that in this case we have for all $p \in U$: The exponential map

$$\exp_p : B(0,\varepsilon) \subseteq \mathrm{T}_pM \to \exp_p(B(0,\varepsilon))$$

is a diffeomorphism onto its image, which contains U. In this context, we now prove the following.

Theorem 2.61. Retain the above notation. In particular, let U be a totally normal neighbourhood of $p_0 \in M$. Then

(i) for all $p, q \in U$ there is a unique geodesic c of length $l(c) < \varepsilon$ connecting p to q, and
(ii) for any piecewise C^1-curve $\gamma : [0,1] \to M$ with $\gamma(0) = p$ and $\gamma(1) = q$ we have $l(\gamma) \geq l(c)$, with equality if and only if γ equals c up to reparametrisation.

The main ingredient in the proof of Theorem 2.61 is a lemma of Gauss which expresses the Riemannian metric in polar coordinates. As preparation, consider $p \in M$ and $\varepsilon > 0$ such that $\exp_p : B := B(0,\varepsilon) \to M$ is a diffeomorphism onto its image B'. Further, let $S^{n-1} := \{v \in \mathrm{T}_pM \mid \|v\| = 1\}$. We introduce polar coordinates on $B\backslash\{0\}$ via

$$\begin{array}{ccc} B\backslash\{0\} & \xrightarrow{\exp_p} & B'\backslash\{p\} \\ \uparrow & \nearrow_{f} & \\ (0,\varepsilon)\times S^{n-1} & & \end{array}$$

where $f(r, v) := \exp_p(rv)$.

Lemma 2.62 (Gauss). Retain the above notation. The decomposition

$$\mathrm{T}_{(r,v)}((0,\varepsilon)\times S^{n-1}) = \mathrm{T}_r(0,\infty) \oplus \mathrm{T}_vS^{n-1}$$

is orthogonal with respect to $f^*(g)_{(r,v)}$. In these coordinates,

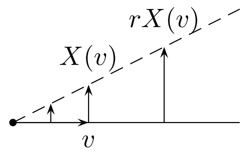
$$f^*(g)_{(r,v)} = dr^2 + h_{(r,v)}$$

where $h_{(r,v)}$ is a scalar product on T_vS^{n-1} depending on r.

Proof. First of all, observe that the restriction to $\mathrm{T}_r(0,\infty)$ of $f^*(g)_{(r,v)}$ equals dr^2. Indeed, the map

$$c_v : (0,\varepsilon) \to B',\ r \mapsto \exp_p(rv) = f(r, v)$$

is a geodesic parametrized by arc length. To show the asserted orthogonality, we consider certain vector fields: Let X be a smooth vector field on S^{n-1}, considered as a map $S^{n-1} \to \mathrm{T}_pM$, $v \mapsto X(v)$ satisfying $\langle v, X(v)\rangle = 0$ for all $v \in S^{n-1}$. It induces the vector field $\widetilde{X}(r,v) := rX(r,v)$ on $B\backslash\{0\}$ via the polar coordinatization.



Furthermore, we let $Y := \exp_{p,*}(\widetilde{X})$ be the direct image of $\widetilde{X}$ via the diffeomorphism f, that is

$$Y(f(r,v)) = D_{(r,v)}f(X(r,v)) = D_{rv}\exp_p(rX(v)).$$

Now, let $\partial/\partial r$ denote the radial vector field on $(0,\varepsilon \times S^{n-1}$.

Then the lemma follows if we show that $f_*(\partial/\partial r)$ and Y are orthogonal at every point $f(r,v)$. To this end, observe that

$$f_*\left(\frac{\partial}{\partial r}\right)(f(r,v)) = f_*\left(\frac{\partial}{\partial r}\right)(c_v(r)) = c_v'(r) \in \mathrm{T}_{f(r,v)}M.$$

We therefore have using Proposition 2.42 and the properties of the Levi-Civita connection

$$\begin{aligned}
\frac{d}{dr}g\left(Y(f(r,v)), f_*\left(\frac{\partial}{\partial r}\right)(f(r,v))\right) &= \frac{d}{dr}g(Y(c_v(r)), c_v'(r)) \\
&= g(\nabla_{c_v'}Y, c_v') + g(Y, \underbrace{\nabla_{c_v'}c_v'}_{=0}) \\
&= g(\nabla_Y c_v', c_v') + g([c_v', Y], c_v') \\
&= \frac{1}{2}Yg(c_v', c_v') + g([c_v', Y], c_v').
\end{aligned}$$

Since c_v is a geodesic, $g(c_v', c_v')_{c_v(t)}$ is constant in t with initial value $g(v,v) = 1$ at $t = 0$. Thus the map $(r,v) \mapsto g(c_v', c_v')_{c_v(r)}$ is constant and hence $Yg(c_v', c_v') = 0$. It therefore remains to evaluate $[c_v', Y]$: We have

$$[c_v', Y] = f_*\left(\left[\frac{\partial}{\partial r}, X\right]\right)$$

A simple computation yields $[\partial/\partial r, X] = 0$. Overall, we conclude

$$\frac{d}{dr}g\left(Y, f_*\left(\frac{\partial}{\partial r}\right)\right) = 0.$$

Since $Y(0) = 0$ this concludes the lemma. □

We now turn to Theorem 2.61

Proof. (Theorem 2.61). Regarding the first assertion, we already know that there is a unique $v \in \mathrm{T}_pM$ with $\|v\| < \varepsilon$ and $c_v(1) = q$. Clearly, $l(c_v([0,1])) = \|v\| < \varepsilon$. Conversely, assume there is $w \in \mathrm{T}_pM$ and $t > 0$ such that $c_w(t) = q$ and $l(c_w([0,t])) < \varepsilon$. Since $l(c_w([0,t])) = t\|w\|$, setting $u := tw$ yields

$$c_u(1) = c_{tw}(1) = c_w(t) = q$$

by homogeneity of geodesics. And since $\|u\| = t\|w\| < \varepsilon$ we conclude $u = v$ by uniqueness of geodesics.

For the second assertion, assume that $\gamma : [0,1] \to M$ is a piecewise C^1-curve with $\gamma(0) = p$ and $\gamma(1) = q$. Then either $\gamma([0,1])$ is contained in B' or there is $s \in (0,1)$

such that $\gamma[0,s) \subseteq B'$ and $\gamma(s) \notin B'$. For $0 \leq t \leq s$, write $\gamma(t) = f(r(t), v(t))$. Then Lemma 2.62 implies

$$\begin{aligned} l(\gamma) &> \int_0^s \sqrt{g_{\gamma(t)}(\gamma'(t), \gamma'(t))}\ dt \\ &\geq \int_0^s |r'(t)|\ dt \geq \left| \int_0^s r'(t)\ dt \right| = \varepsilon. \end{aligned}$$

Therefore, $l(\gamma)) > \varepsilon > l(c)$. Suppose now that $\gamma([0,1])$ is contained in B'. Again, using polar coordinates, we have

$$l(\gamma) = \int_0^1 \sqrt{[r'(t)^2 + h_{(r(t),v(t))}(v'(t), v'(t))]}\ dt \geq r(1) - r(0) = l(c).$$

Moreover, equality occurs if and only if $h_{(r(t),v(t))}(v'(t), v'(t)) = 0$ for all $0 \leq t \leq 1$, i.e. $v'(t) = 0$ whence $v(t) = v$ for all $t \in [0,1]$, and r is monotone. Assuming that γ is parametrized proportional to arc length with velocity l, we get $\gamma(t) = f(lt, v)$ and $\gamma(t/l) = c_v(t)$. □

Corollary 2.63. Retain the above notation. A piecewise C^1-curve $\gamma : I \to M$ parametrized proportional to arc length is a geodesic if and only if for all $t \in I$ there is $\varepsilon > 0$ such that $d(c(t-\varepsilon), c(t+\varepsilon)) = l(c([t-\varepsilon, t+\varepsilon]))$.

Proof. This is due to Theorem 2.61 if γ is C^1 and is left as an exercise otherwise. □

Remark 2.64. As mentioned in the introduction, the examples of the sphere and the torus show that in general geodesics are not globally length minimizing.

Definition 2.65. Retain the above notation. A geodesic $c : [a,b] \to M$ is *minimal* if $l(c([a,b])) = d(c(a), c(b))$.

Proposition 2.66. Retain the above notation. Let $c : [a,b] \to M$ be piecewise C^1. If $l(c) = d(c(a), c(b))$ and c is parametrized proportional to arc length then c is a geodesic.

Proof. Observe that if $a \leq a' \leq b' \leq b$ then $l(c([a',b'])) = d(c(a'), c(b'))$. Indeed, otherwise $l(c([a',b'])) > d(c(a'), c(b'))$ and there is by definition a piecewise C^1-curve $\gamma : [a',b'] \to M$ with $l(\gamma) < l(c([a',b']))$, $\gamma(a') = c(a')$ and $\gamma(b') = c(b')$. But then we have for the concatenation η of $c|_{[a,a']}$, γ and $c|_{[b',b]}$:

$$\begin{aligned} l(\eta) &= l(c([a,a'])) + l(\gamma) + l(c([b',b])) \\ &= l(c) - l(c([a',b'])) + l(\gamma) \\ &< l(c) = d(c(a), c(b)). \end{aligned}$$

Combining this observation with Corollary 2.63 implies the proposition. □

Normal Coordinates. As before, let (M,g) be a Riemannian manifold with Levi-Civita connection ∇. Recall that in a local coordinate system (U,φ) of M we have defined the Christoffel symbols Γ^i_{jk} by

$$\nabla_{\partial_j}\partial_k = \sum_i \Gamma^i_{j,k}\partial_i.$$

Since $\nabla_{\partial_j}\partial_k - \nabla_{\partial_k}\partial_j = [\partial_j, \partial_k] = 0$ we get $\nabla_{\partial_k}\partial_j = \sum_i \Gamma^i_{kj}\partial_i = \sum_i \Gamma^i_{jk}\partial_i$ and hence $\Gamma^i_{kj} = \Gamma^i_{jk}$.

Now let $p \in M$ and $\varepsilon > 0$ such that $\exp_p : B(0,\varepsilon) \to U \subseteq M$ is a diffeomorphism onto an open neighbourhood U of p. Choosing an orthonormal basis $e_1, \dots, e_m$ of (T_pM, g_p) and setting $\varphi := \exp_p^{-1} : U \to \mathrm{T}_pM$, we obtain local coordinates at $p \in M$. In these coordinates we have

(i) $g_p(\partial_i(p), \partial_j(p)) = \delta_{ij}$, and

(ii) $\nabla_{\partial_i}\partial_j(p) = 0$

Indeed, for (i) note that by definition $D_0 \exp_p(e_i) = \partial_i(p)$ and since $D_0 \exp_p = \mathrm{Id}$ we get $g_p(\partial_i(p), \partial_j(p)) = g_p(e_i, e_j) = \delta_{ij}$. For the second assertion, let $v \in \mathrm{T}_pM$ and $c_v : I \to M$, $t \mapsto c_v(t)$ the corresponding geodesics. Then by definition $\exp_p^{-1}(c_v(t)) = tv$, i.e. $\varphi(c_v(t)) = (x_1(t), \dots, x_m(t))$ with $x_i(t) = tv_i$. Since c_v is a geodesic we have for all $i \in \{1, \dots, m\}$:

$$x_i''(t) + \sum_{j,k} \Gamma^i_{kj}(c_v(t))x_k'(t)x_j'(t) = 0.$$

Evaluating at $t = 0$ yields

$$\sum_{k,j} \Gamma^i_{kj}(p)v_k v_j = 0.$$

Since Γ^i_{kj} is symmetric in (k, j) and the above holds for any vector $v \in B(0, \varepsilon)$ we conclude $\Gamma^i_{kj}(p) = 0$.

2.6. **The Hopf-Rinow Theorem.** Let (M, g) be a connected Riemannian manifold. We have seen that one can define a distance on M by setting

$$d(p, q) := \inf\{l(c) \mid c : [0, 1] \to M \text{ is piecewise } C^1, c(0) = p, c(1) = q\}$$

for all $p, q \in M$. In addition, d induces the given topology on M.

Definition 2.67. A Riemannian manifold (M, g) is *geodesically complete* if for every $p \in M$, the expoential map $\exp_p$ is defined on the whole of T_pM.

That is, in the case of geodesically complete Riemannian manifold M, given $(p, v) \in \mathrm{T}M$ the geodesic $c_{(p,v)}(t)$ is defined for all $t \in \mathbb{R}$. In this section, we prove the following fundamental theorem, linking the properties of (M, d) as a metric space with geodesic completeness.

Theorem 2.68 (Hopf-Rinow). Let (M, g) be a Riemannian manifold, $p \in M$ and d the metric induced by g. Then the following statements are equivalent.

(i) The map $\exp_p$ is defined on the whole of T_pM.
(ii) Closed and bounded sets in M are compact.
(iii) The metric space (M, d) is complete.
(iv) The Riemannian manifold (M, g) is geodesically complete.

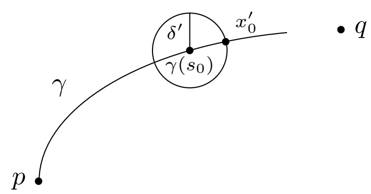
In addition, any of the above statements implies the following.

(v) For any $q \in M$ there is a minimal geodesic joining p to q.

Proof. The main point of the proof lies in the implication (i)⇒(v). For this, we first need to find a candidate for the tangent vector $v \in \mathrm{T}_pM$ giving the minimal geodesic. To this end, let $\delta > 0$ be such that $B(p, \delta)$ is a normal ball, that is, the exponential map $\exp_p : B(0, \delta) \subseteq \mathrm{T}_pM \to M$ is a diffeomorphism onto its image. Let $S := \exp(S(0, \delta))$ be the image of the boundary of $B(0, \delta)$ and let $x_0 \in S$ be a point such that $d(x_0, q) = \min\{d(x, q) \mid x \in S\}$ which exists by compactness of S. Now, let $v \in \mathrm{T}_pM$ be the unique vector with $\|v\| = 1$ and $\exp_p(\delta v) = x_0$, and let $\gamma : \mathbb{R} \to M$, $s \mapsto \exp_p(sv)$ be the corresponding geodesic which by assumption is defined on the whole of $\mathbb{R}$. We proceed to show that $\gamma(r) = q$ for $r := d(p, q)$: Consider the set $A := \{s \in [0, r] \mid d(\gamma(s), q) = r - s\}$. Then A is non-empty since $0 \in A$ and closed by continuity. Now, let $s_0 \in A$ with $s_0 < r$ and $\delta' > 0$ such that

(i) $s_0 + \delta' \leq r$, and
(ii) $B(\gamma(s_0), \delta')$ is a normal ball.

In this setting, we show that $s_0+\delta' \in A$. Since A is closed, this will imply that $r \in A$ which concludes this part of the proof. Consider the following figure.



Let $x_0' \in S' := \exp_{\gamma(s_0)}(B(0,\delta'))$ be such that $d(x_0',q) = \min\{d(x,q) \mid x \in S'\}$. We claim that $x_0' = \gamma(s_0+\delta')$: Let $v' \in \mathrm{T}_{\gamma(s_0)}M$ with $\|v'\| = 1$ be such that $\exp_{\gamma(s_0)} \delta' v' = x_0'$. The concatenation η of the geodesic $\gamma([0,s_0])$ with $\gamma'([0,\delta'])$ where $\gamma'(t) = \exp_{\gamma(s_0)}(tv')$ for $0 \le t \le \delta'$ is a piecewise C^1-curve joining p to x_0' and has length $l(\eta) = s_0+\delta'$. On the other hand, we have

$$d(p,x_0') \ge d(p,q) - d(q,x_0') \tag{1}$$

and

$$d(\gamma(s_0),q) = \delta' + \min\{d(x,q) \mid x \in S'\} = \delta' + d(x_0',q). \tag{2}$$

Equation (2) implies

$$r - s_0 = \delta' + d(x_0',q). \tag{3}$$

Substituting (3) back into (1) yields

$$d(p,x_0') \ge r - (r - s_0 - \delta') = s_0 + \delta'.$$

Since $l(\eta) = s_0+\delta'$ we conclude that $d(p,x_0') = s_0+\delta'$ and that η is a minimal geodesic joining p to x_0'. Given that $\eta'(0) = \gamma'(0)$ we conclude that $\eta(t) = \gamma(t)$ for all $t \in [0, s_0+\delta']$. In particular, $\gamma(s_0+\delta') = x_0'$ which proves the claim.

We proceeed by showing how the claim implies $s_0+\delta' \in A$: Indeed, (3) now implies

$$r - s_0 = \delta' + d(x_0',q) = \delta' + d(\gamma(s_0+\delta'),q).$$

Equivalently, $d(\gamma(s_0+\delta'),q) = r - (s_0+\delta')$, i.e. $s_0+\delta' \in A$.

For the implication (i)⇒(ii), let F be a closed and bounded set. Then

$$\sup\{d(p,x) \mid x \in F\} < \infty.$$

We may hence choose $T > 0$ with $d(p,x) \le T$ for all $x \in F$. By (v), we have $\exp_p(\overline{B(0,T)}) \supseteq F$. That is, F is a closed subset of a compact set and hence compact itself.

The implication (ii)⇒(iii) is a general topology statement.

To prove that (iii) implies (iv), assume that (M,g) is not geodesically complete. Then there is $q \in M$ and $v \in \mathrm{T}_qM$ with $\|v\| = 1$ such that $\gamma(s) := \exp_q(sv)$ is defined on $[0,s_0)$ but not for $s = s_0$. Pick a sequence $(s_n)_{n\in\mathbb{N}}$ with $s_n < s_0$ for all $n \in \mathbb{N}$ such that $\lim_{n\to\infty} s_n = s_0$. Then $d(\gamma(s_n),\gamma(s_m)) \le |s_n - s_m|$ and hence $(\gamma(s_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in M, which by assumption converges to a point, say $p_0 = \lim_{n\to\infty}\gamma(s_n)$. Let (W,δ) be a totally normal neighbourhood of p_0. That is, for all $p \in W$, the map

$$\exp_p : B(0,\delta) \to \exp_p(B(0,\delta))$$

is a diffeomorphism and $\exp_p(B(0,\delta)) \supseteq W$. For large enough n,m, the images points $\gamma(s_n)$ and $\gamma(s_m)$ lie in W and $|s_n - s_m| < \delta$. Now there is a unique geodesic

g joining $\gamma(s_n)$ to $\gamma(s_m)$ of length strictly less than δ. Clearly, γ coincides with g whenever defined. Now $\exp_{\gamma(s_n)} : B(0,\delta) \to M$ is a diffeomorphism onto its image which contains W. Hence g extends γ beyond s_0. □

Corollary 2.69. Every compact connected Riemannian manifold ist geodesically complete.

Corollary 2.70. Let G be a compact connected Lie group. Then the Lie group exponential $\exp_G : \mathrm{Lie}(G) \to G$ is surjective.

Corollary 2.71. Every closed submanifold of a complete connected Riemannian manifold is complete.

3. Curvature

In this section we finally discuss various locally defined notions of curvature and their global implications. The first was introduced by Gauss for surfaces in $\mathbb{R}^3$ and later on generalized by Riemann to sectional curvature: Given a Riemannian manifold (M,g) and a point $p \in M$, look at two-dimensional subspaces E of T_pM and apply Gauss' notion of curvature to the surface pieces that arise from looking at small geodesics with initial velocity in E. This yields a local notion of curvature on M depending on the choice of a tangent plane.

There is a notion of curvature which subsumes all the above and many others, called Riemannian curvature. Although not invented by Riemann himself it is determined by all the sectional curvatures above.

3.1. **Definition and Formal Properties.** Given a Riemannian manifold (M,g) with Levi-Civita connection ∇, consider the map

$$R(X,Y) : \Gamma(\mathrm{T}M) \to \Gamma(\mathrm{T}M),\ Z \mapsto \nabla_Y\nabla_X Z - \nabla_X\nabla_Y Z + \nabla_{[X,Y]}Z$$

One way to at it is to study the extent to which the map $\Gamma(\mathrm{T}M) \to \mathrm{End}(\Gamma(\mathrm{T}M))$ which maps X to ∇_X fails to be a Lie algebra homomorphism.

Proposition 3.1. Retain the above notation. The map

$$\Gamma(\mathrm{T}M) \times \Gamma(\mathrm{T}M) \times \Gamma(\mathrm{T}M) \to \Gamma(\mathrm{T}M),\ (X,Y,Z) \mapsto R(X,Y)Z$$

is a tri-linear map of $C^\infty(M)$-modules.

Proof. As prepration for the proof, we compute for $f, \varphi \in C^\infty(M)$:

$$\begin{aligned}[fX,Y](\varphi) &= (fX)(Y(\varphi)) - Y(fX)(\varphi)\\ &= fX(Y(\varphi)) - Y(f)Y(\varphi) - fYX(\varphi) = (f[X,Y] - Y(f)X)(\varphi).\end{aligned}$$

Therefore, $[fX,Y] = f[X,Y] - Y(f)X$ and $[X,gY] = g[X,Y] + X(g)Y$. Now, we treat the $C^\infty(M)$-linearity of the given map in its three slots individually. For the first one, we have

$$\begin{aligned}R(fX,Y)Z &= \nabla_Y\nabla_{fX}Z - \nabla_{fX}\nabla_Y Z + \nabla_{[fX,Y]}Z\\ &= \nabla_Y(f\nabla_X Z) - f\nabla_X\nabla_Y Z + f\nabla_{[X,Y]}Z - Y(f)\nabla_X Z\\ &= Y(f)\nabla_X Z + f(\nabla_Y\nabla_X Z - \nabla_X\nabla_Y Z + \nabla_{[X,Y]}Z) - Y(f)\nabla_X Z\\ &= fR(X,Y)Z.\end{aligned}$$

A similar computation works in the case of the second variable. Concerning the Z-variable, first compute

$$\begin{aligned}\nabla_Y\nabla_X(fZ) &= \nabla_Y(X(f)Z + f\nabla_X Z)\\ &= Y(X(f))Z + X(f)\nabla_Y Z + Y(f)\nabla_X Z + f\nabla_Y\nabla_X Z\end{aligned}$$

and similarly $\nabla_X\nabla_Y(fZ)$. The difference of the two equals

$$\begin{aligned} &Y(X(f))Z + f\nabla_Y\nabla_X Z - X(Y(f))Z - f\nabla_X\nabla_Y Z \\ &\qquad = (Y(X(f)) - X(Y(f)))Z + f(\nabla_Y\nabla_X Z - \nabla_X\nabla_Y Z). \end{aligned}$$

On the other hand, we have

$$\nabla_{[X,Y]} fZ = [X,Y](f)Z + f\nabla_{[X,Y]}Z = (X(Y(f)) - Y(X(f)))Z + f\nabla_{[X,Y]}Z.$$

Overall, this completes the proof. □

Now, let (U,φ) be a chart of M at $p\in M$ and let $(\partial_i)_{i=1}^m$ be the associated coordinate vector fields on U. Then

$$R(\partial_i,\partial_j)\partial_k = \sum_l R_{ijk}^l \partial_l$$

where $R_{ijk}^l : U \to \mathbb{R}$ are smooth functions on U. Decomposing $X = \sum_i X_i\partial_i$, $Y = \sum_j Y_j\partial_j$ and $Z = \sum_k Z_k\partial_k$ on U we get

$$\begin{aligned} R(X,Y)Z &= R\left(\sum_i X_i\partial_i, \sum_j Y_j\partial_j\right)\left(\sum_k Z_k\partial_k\right) \\ &= \sum_{i,j,k} X_iY_jZ_k R(\partial_i,\partial_j)\partial_k \\ &= \sum_{i,j,k,l} X_iY_jZ_k R_{ijk}^l\partial_l. \end{aligned}$$

As a corollary we record that $R(X,Y)Z \in \Gamma(\mathrm{T}M)$ is a tensor on M in the following sense.

Corollary 3.2. Retain the above notation. The value of $R(X,Y)Z(p)$ only depends on $X(p)$, $Y(p)$ and $Z(p)$.

Definition 3.3. Let (M,g) be a Riemannian manifold with Levi-Civita connection ∇. The *Riemannian curvature tensor* of (M,g) is the collection of trilinear maps

$$R_p : \mathrm{T}_pM \times \mathrm{T}_pM \times \mathrm{T}_pM \to \mathrm{T}_pM,\ (v,w,u) \mapsto R(X,Y)Z(p) \quad (p\in M)$$

where X,Y,Z are vector fields defined on open neighbourhoods of $p\in M$ with $X(p)=v)$, $Y(p)=w$ and $Z(p)=u$.

Equivalently, one can view the Riemannian curvature as a collection $R_p(v,w) \in \mathrm{End}(\mathrm{T}_pM)$ of endomorphisms, $v,w\in \mathrm{T}_pM$, $p\in M$.

All curvature notions that we discuss are descendents of the Riemannian curvature tensor but may have a more explicit geometric interpretation. In order to work with these, we establish certain symmetry properties of R, coming from the Lie algbera structure on $\Gamma(\mathrm{T}M)$ as well as the relation between the Levi-Civita connection and the metric.

Proposition 3.4 (Bianchi). Retain the above notation, let $X,Y,Z\in\Gamma(\mathrm{T}M)$. Then

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0.$$

Proof. Expanding the definition, the proposition amounts to proving

$$\begin{aligned} 0 &= \nabla_Y\nabla_X Z - \nabla_X\nabla_Y Z + \nabla_{[X,Y]}Z \\ &\quad + \nabla_Z\nabla_Y X - \nabla_Y\nabla_Z X + \nabla_{[Y,Z]}X \\ &\quad + \nabla_X\nabla_Z Y - \nabla_Z\nabla_X Y + \nabla_{[Z,X]}Y. \end{aligned}$$

Pairing appropriate expressions, this retranslates to

$$0 = \nabla_Y(\nabla_X Z - \nabla_Z X) + \nabla_X(\nabla_Z Y - \nabla_Y Z) + \nabla_Z(\nabla_Y X - \nabla_X Y) \\ + \nabla_{[X,Y]}Z + \nabla_{[Y,Z]}X + \nabla_{[Z,Y]}Y$$

Using relations like $\nabla_X Z - \nabla_Z X = [X, Z]$ the above reduces to the Jacobi identity. □

For the following, consider $X, Y, Z, T \in \Gamma(\mathrm{T}M)$ on M and define for $p \in M$:

$$(X, Y, Z, T)_p := \langle R(X, Y)Z(p), T(p)\rangle_{\mathrm{T}_pM}.$$

This way, we hope to deduce further properties of R taking into account the fact ∇ is not just a connection but behaves nicely with respect to the metric.

Proposition 3.5. Retain the above notation. Then the following hold.

(i) $(X, Y, Z, T) + (Y, Z, X, T) + (Z, X, Y, T) = 0$.
(ii) $(X, Y, Z, T) = -(Y, X, Z, T)$.
(iii) $(X, Y, Z, T) = -(X, Y, T, Z)$.
(iv) $(X, Y, Z, T) = (Z, T, X, Y)$.

A way to remember Proposition 3.5 is to note the simple behaviour of the form $(-, -, -, -)$ under the action of the subgroup of S_4 which preserves the block decomposition $\{\{1, 2\}, \{3, 4\}\}$ of $\{1, 2, 3, 4\}$ depicted by (ii), (iii) and (iv). If one is concerned with oher permutations one has to use Bianchi's equality (i).

Proof. (Proposition 3.5). As remarked above, (i) is merely a reformulation of Bianchi's identity. The second assertion is an immediate consequnce of the definition. For (iii), note that by doubling variables, it suffices to show that $(X, Y, Z, Z) = 0$. We are therefore looking at $\langle \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]}Z - Z\rangle$. Using that ∇ is the Levi-Civita connection, we have

$$\langle \nabla_Y \nabla_X Z, Z\rangle = Y\langle \nabla_X Z, Z\rangle - \langle \nabla_X Z, \nabla_Y Z\rangle$$

and

$$\langle \nabla_X \nabla_Y Z, Z\rangle = X\langle \nabla_Y Z, Z\rangle - \langle \nabla_Y Z, \nabla_X Z\rangle.$$

Taking the difference of the above equalities we obtain $Y\langle \nabla_X Z, Z\rangle - X\langle \nabla_Y Z, Z\rangle$. Observing that

$$\langle Z, Z\rangle = \langle \nabla_X Z, Z\rangle + \langle Z, \nabla_X Z\rangle = 2\langle \nabla_X Z, Z\rangle$$

the above difference is

$$\frac{1}{2}YX\langle Z, Z\rangle - \frac{1}{2}XY\langle Z, Z\rangle = -\frac{1}{2}[X, Y]\langle Z, Z\rangle = \langle \nabla_{[X,Y]}Z, Z\rangle.$$

Hence the assertion. Finally, for (iv) compute all instances of Bianchi's identity (i) arising from cyclic permutation of the variables and sum the result. Due to (ii) and (iii), one obtains

$$2(X, Z, T, Y) - 2(T, Y, X, Z) = 0$$

which is (iv). □

If so inclined, one can make the functions $R^l_{ijk}\partial_l$ occuring in the equality

$$R(\partial_i, \partial_j)\partial_k = \sum_l R^l_{ijk}\partial_l$$

explicit in terms of the metric as in the case of the Christoffel symbols: Given that $[\partial_i, \partial_j] = 0$ for all $i, j \in \{1, \dots, m\}$ we have $R(\partial_i, \partial_j)\partial_k = \nabla_{\partial_j}\nabla_{\partial_i}\partial_k - \nabla_{\partial_i}\nabla_{\partial_j}\partial_k$.

Recall that $\nabla_{\partial_i}\partial_k = \sum_l \Gamma^l_{ik}\partial_l$. Plugging this into the above and comparing coefficients one gets the following, seldomly used, formula

$$R^l_{ijk} = (\partial_j\Gamma^l_{ik} - \partial_i\Gamma^l_{jk}) + \sum_s \left(\Gamma^s_{ik}\Gamma^l_{js} - \Gamma^s_{jk}\Gamma^l_{is}\right)$$

Going back to the formula of the Christoffel symbols in terms of the Riemannian metric, one obtains an expression of R^l_{ijk} in terms of the same.

3.1.1. *Sectional Curvature.* We now turn to sectional curvature, introduced by Riemann, which is simpler than the Riemannian curvature tensor above but still determines it. Given vectors x and y in a Euclidean space, let $|x\wedge y| = \sqrt{|x|^2|y|^2 - \langle x,y\rangle^2}$ denote the area of the parallelogram spanned by x and y.

Proposition 3.6. Let (M,g) be a Riemannian manifold, $p \in M$ and $E \leq \mathrm{T}_pM$ two-dimensional. Given a basis (x,y) of E, the quantity

$$K(x,y) := \frac{(x,y,x,y)}{|x\wedge y|^2}$$

does not depend on the choice of basis of E.

Definition 3.7. Retain the above notation. The *sectional curvature of M at p with respect to E* is given by $K(E) := K(x,y)$ where (x,y) is any basis of E.

Proof. We show that K is invariant under the following transformations:
(i) $(x,y) \mapsto (x,y)$,
(ii) $(x,y) \mapsto (\lambda x, y)$, $\lambda \in \mathbb{R}\setminus\{0\}$,
(iii) $(x,y) \mapsto (x, y+\mu x)$, $\mu \in \mathbb{R}$.

This suffices since the associated matrices

$$\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \begin{pmatrix} \lambda & \\ & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & \mu \\ & 1 \end{pmatrix}$$

generate $\mathrm{GL}(2,\mathbb{R})$ for the given ranges of λ and μ. In particular, they realize any possible change of basis. For the first transformation, note that

$$(y,x,y,x) = -(x,y,y,x) = (x,y,x,y)$$

and $|x\wedge y|^2 = |y\wedge x|^2$. For (ii), observe

$$(\lambda x, y, \lambda x, y) = \lambda^2(x,y,x,y)$$

and $|\lambda x\wedge y|^2 = \lambda^2|x\wedge y|^2$. Finally, for the third assertion, compute

$$\begin{aligned}(x, y+\mu x, x, y+\mu x) &= (x,y,x,y) + (x,y,x,\mu x) + (x,\mu x,x,y) + (x,\mu x,x,\mu x)\\ &= (x,y,x,y) + \mu(x,y,x,x) + \mu(x,x,x,y) + \mu^2(x,x,x,x)\\ &= (x,y,x,y)\end{aligned}$$

and $|x\wedge(y+\mu x)|^2 = |x\wedge y|^2$. □

Remark 3.8. The Riemannian curvature tensor can be recovered from sectional curvature in the following way: For $x,y,z,t \in \mathrm{T}_pM$ we have

$$\frac{\partial^2}{\partial\alpha\partial\beta}(x+\alpha z, y+\beta t, x+\alpha z, y+\beta t) = 6(x,y,z,t)$$

In order to give some examples, we make the following preliminary remarks: Let (M,g) be a Riemannian manifold and $f \in \mathrm{Iso}(M,g)$. Using the uniqueness of the Levi-Civita connection or the formula for the same developed in the proof one verifies that for any $X,Y,Z \in \mathrm{T}M$ we have

$$\nabla_{f_*X} f_*Y = f_*(\nabla_X Y)$$

from which we deduce $R(f_*X, f_*Y)f_*Z = f_*(R(X,Y)Z)$ by definition of R. Hence for $x \in M$ and $E \leq \mathrm{T}_x M$ two-dimensional we have $K_x(E) = K_{f(x)}(D_x f(E))$. In particular, the map

$$K_x : \mathrm{Gr}_2(\mathrm{T}_x M) \to \mathbb{R}$$

is invariant under $\mathrm{stab}_{\mathrm{Iso}(M,g)}(x)$.

Example 3.9. The above remarks facilitate the computation of sectional curvature enormously in the following examples.

(i) The Riemannian curvature of $(\mathbb{R}^n, \mathrm{can})$ vanishes identically: Indeed, for $X, Y \in \mathrm{T}\,\mathbb{R}^n$ we have $\nabla_X Y = L_X Y$ if Y is considered as a map from $\mathbb{R}^n$ to $\mathbb{R}^n$. Consequently,

$$\begin{aligned} R(X,Y)Z &= \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z \\ &= L_Y L_X Z - L_X L_Y Z + L_{[X,Y]} Z = 0 \end{aligned}$$

(ii) Now consider $S^n \subseteq \mathbb{R}^{n+1}$ with the induced Riemannian metric. We show that $K : \mathrm{Gr}_2(S^n) \to \mathbb{R}$ is constant: Recall that $\mathrm{O}(n+1)$ acts transtively by isometries on S^n since it does so on $\mathbb{R}^{n+1}$. Next,

$$\mathrm{stab}_{\mathrm{O}(n)}(e_{n+1}) = \left\{ \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \middle| \, A \in \mathrm{O}(n) \right\}$$

acts on $\mathrm{T}_{e_{n+1}} S^n = \{(v,0)^T \in \mathbb{R}^{n+1} \mid v \in \mathbb{R}^n\}$. As remarked above, the map $K_{e_{n+1}} : \mathrm{Gr}_2(\mathrm{T}_{n+1}\,\mathbb{R}^n) \to \mathbb{R}$ is invariant under $\mathrm{stab}_{\mathrm{O}(n+1)}(e_{n+1})$. Observing that the action of $\mathrm{O}(n)$ on $\mathrm{Gr}_2(\mathbb{R}^n)$ is transitive we deduce that $K_{e_{n+1}}$ is constant. Hence so is K by transitivity of $\mathrm{O}(n+1)$.

(iii) Finally, consider hyperbolic space

$$\mathbb{H}^n = \{x \in \mathbb{R}^{n+1} \mid x_1^2 + \cdots + x_n^2 - x_{n+1}^2 = -1,\ x_{n+1} > 0\}.$$

We know that $\mathrm{SO}_0(n,1) = \{g \in \mathrm{O}(n,1) \mid \det g = 1,\ g(\mathbb{H}^n) = \mathbb{H}^n\}$ acts transitively on $\mathbb{H}^n$ by Riemannian isometries. As before, we look at the base point $e_{n+1} \in \mathbb{H}^n$: We have $\mathrm{T}_{e_{n+1}}\,\mathbb{H}^n = \{(v,0)^T \in \mathbb{R}^{n+1} \mid v \in \mathbb{R}^n\}$ and the stabilizer

$$\mathrm{stab}_{\mathrm{SO}_0(n,1)}(e_{n+1}) = \left\{ \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \middle| \, A \in \mathrm{SO}(n) \right\}.$$

The action of this stabilizer on $\mathrm{T}_{e_{n+1}}\,\mathbb{H}^n$ is equivalent to the $\mathrm{SO}(n)$-action on $\mathbb{R}^n$ which is again transitive on two-dimensional subspaces of $\mathbb{R}^{n+1}$. Hence, as before, the sectional curvature of $\mathbb{H}^n$ is constant.

As a matter of fact, the three examples above are the only complete, simply connected Riemannian manifolds with constant sectional curvature.

Example 3.10. As one might expect, things turn out particularly nice in the setting of Lie groups: Let G be a Lie group which admits a bi-invariant Riemannian metric. We have seen that for $X, Y \in \Gamma^{\mathrm{inv}}(\mathrm{T}G)$ we have $\nabla_X Y = \frac{1}{2}[X,Y]$. Hence we obtain for $X, Y, Z \in \Gamma^{\mathrm{inv}}(\mathrm{T}G)$:

$$\begin{aligned} R(X,Y)Z &= \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z \\ &= \frac{1}{4}[Y,[X,Z]] - \frac{1}{4}[X,[Y,Z]] + \frac{1}{2}[[X,Y],Z] \\ &= \frac{1}{4}[[X,Y],Z] \end{aligned}$$

by Jacobi's identity. Notice that since R is a tensor we do not have to worry about only knowing its values on invariant vector fields: For $x, y, z \in \mathfrak{g} = \mathrm{T}_e G$ we have $R_e(x,y)z = \frac{1}{4}[[x,y],z]$.

Now suppose that $x, y \in \mathfrak{g}$ are orthonormal. Then

$$(x,y,x,y) = \left\langle \frac{1}{4}[[x,y],x], y \right\rangle_{\mathfrak{g}} = -\frac{1}{4}\langle [x,[x,y]], y\rangle_{\mathfrak{g}}.$$

Applying the invariance of $\langle -, - \rangle$ under the adjoint representation, i.e. for all $v, w, u \in \mathfrak{g}$ we have

$$\begin{aligned}
0 &= \frac{d}{dt}\bigg|_{t=0} \langle \mathrm{Ad}(\exp(tv))w, \mathrm{Ad}(\exp(tv))u\rangle \\
&= \left\langle 0 = \frac{d}{dt}\bigg|_{t=0} \mathrm{Ad}(\exp(tv))w, v \right\rangle + \left\langle w, \frac{d}{dt}\bigg|_{t=0} \mathrm{Ad}(\exp(tv))u \right\rangle \\
&= \langle [v,w], u\rangle + \langle w, [v,u]\rangle,
\end{aligned}$$

we obtain

$$(x,y,x,y) = \frac{1}{4}\langle [x,y],[x,y]\rangle = \frac{1}{4}\|[x,y]\|^2.$$

In particular, the sectional curvature of any two-dimensional subspace $E \subseteq \mathfrak{g}$ is greater or equal to zero. Equality holds if and only if E is an abelian subalgebra. These considerations lead to root systems of Lie algebras and flat subspaces of manifolds.

3.1.2. *Ricci Curvature and Scalar Curvature.* Although sectional curvature is easier to deal with than the Riemannian curvature tensor, it is still fairly complicated given that it determines the latter. Two weaker and simpler curvature notions are *Ricci curvature* and *scalar curvature* which we define in this section.

Let (M, g) be a Riemannian manifold, $p \in M$ and $x \in \mathrm{T}_pM$ a unit tangent vector. Pick an orthonormal basis $(z_1, \dots, z_m)$ of T_pM such that $z_m = x$.

Definition 3.11. Retain the above notation. Then the *Ricci curvature* of M at p with respect to x is

$$\mathrm{Ric}_p(x) := \frac{1}{m-1}\sum_{i=1}^{m-1} K(x, z_i)$$

and the scalar curvature of M at p is

$$K(p) := \frac{1}{m}\sum_{i=1}^{m} \mathrm{Ric}_p(z_i)$$

Notice that whereas sectional curvature depends on a point and a two-dimensional subspace of that point's tangent space, Ricci curvature only requires a point and a tangent vector, and scalar curvature only depends on a point. We need to show that the above definitions do not depend on the choice of orthonormal basis involving the given vector $x \in \mathrm{T}_pM$. To this end, consider the bilinear map

$$Q : \mathrm{T}_pM \times \mathrm{T}_pM \to \mathbb{R},\ (x,y) \mapsto \mathrm{tr}(z \mapsto R_p(x,z)y).$$

Then Q is symmetric since

$$Q(x,y) = \sum_{i=1}^{m}\langle R_p(x,z_i)y, z_i\rangle = \sum_{i=1}^{m}(x,z_i,y,z_i) = \sum_{i=1}^{m}(y,z_i,x,z_i) = Q(y,x).$$

Thus Q is a symmetric bilinear form and

$$Q(x,x) = \sum_{i=1}^{m}(x,z_i,x,z_i) = \sum_{i=1}^{m-1}(x,z_i,x,z_i) = \sum_{i=1}^{m-1} K(x,z_i) = (m-1)\mathrm{Ric}_p(x).$$

For scalar curvature, let $K \in \mathrm{End}(\mathrm{T}_pM)$ be the symmetric endomorphism with $Q(x, y) = \langle K(x), y\rangle$. Then

$$\mathrm{tr}(K) = \sum_{i=1}^{m} \langle K(z_i), z_i\rangle = \sum_{i=1}^{m} Q(z_i, z_i) = (m-1)\sum_{i=1}^{m} \mathrm{Ric}_p(z_i) = m(m-1)K(p).$$

3.2. **The first and second variation formula.** We now aim to deduce global properties of a manifold from local curvature properties. To this end, we first need to generalize the notions "vector field along a curve" and "covariant derivative" to the setting of "parametrized submanifolds". Let N and M be a manifolds and let $h : N \to M$ be a smooth map. In practice, N is a domain in $\mathbb{R}^2$. Recall that

$$h^*(\mathrm{T}M) := \{(x, v) \in N \times \mathrm{T}M \mid v \in \mathrm{T}_{h(x)}M\}$$

is the pullback bundle of $\mathrm{T}M$ under h. Its sections are *vector fields along h*. The following two Lie algebra homomorphisms come with the above definition:

$$\Gamma(\mathrm{T}N) \to \Gamma(h^*\mathrm{T}M),\ X \mapsto \overline{X},\ \overline{X}(x) := D_xh(X(x)),\ \text{and}$$
$$\Gamma(\mathrm{T}M) \to \Gamma(h^*\mathrm{T}M),\ Y \mapsto h^*(Y),\ h^*(Y)(x) = Y(h(x)).$$

The following generalizes Theorem 2.37.

Proposition 3.12. Retain the above notation. There is a bilinear map

$$\nabla^h : \Gamma(\mathrm{T}N) \times \Gamma(h^*(\mathrm{T}M)) \to \Gamma(h^*(\mathrm{T}M))$$

such that for all $X \in \Gamma(\mathrm{T}N)$, $Y \in \Gamma(h^*(\mathrm{T}M))$ and $f \in C^\infty(N)$:

(i) $\nabla^h_{fX}Y = f\nabla^h_X Y$, and
(ii) $\nabla^h_X(fY) = X(f)Y + f\nabla^h_X Y$.

Furthermore, we have for $X \in \Gamma(\mathrm{T}N)$ and $Y \in \Gamma(\mathrm{T}M)$:

(iii) $\nabla^h_X h^*(Y) = h^*(\nabla_{\overline{X}}Y)$.

Proof. We proceed as in the case of the covariant derivative along curves. If (U, φ) is a chart of M and $V := h^{-1}(U)$ then for $Y \in \Gamma(h^*(\mathrm{T}M))$ and $y \in V$ we have

$$Y(y) = \sum_{i=1}^{m} Y_i(y)h^*(\partial_i)(y).$$

Assuming that ∇^h with the asserted properties exists we thus have

$$\nabla^h_X Y(y) = \sum_{i=1}^{m} X(Y_i)(y)\partial_i(h(y)) + \sum_{i=1}^{m} Y_i(y)\left(\nabla^h_X\partial_i\right)(h(y))$$

Using

$$(\nabla^h_X\partial_i)(h(y)) = h^*\left(\nabla_{\overline{X}}\partial_i\right)(y) = \left(\nabla_{D_yh(X(y))}\partial_i\right)(h(y))$$

we thus have

$$\nabla^h_X Y(y) = \sum_{i=1}^{m} X(Y_i)(y)\partial_i(h(y)) + \sum_{i=1}^{m} Y_i(y)\left(\nabla_{D_yh(X(y))}\partial_i\right)(h(y))$$

which shows uniqueness. As before, the above can be used to define $\nabla^h_X Y$ locally from which one deduces the asserted properties. □

One can now pretend that ∇^h produces a curavture notion and obtain the following.

Proposition 3.13. Let M and N be Riemannian manifolds and let $h : N \to M$ be a smooth map. Further, let $X, Y \in \Gamma(\mathrm{T}N)$ and $U, V \in \Gamma(h^*\mathrm{T}M)$. Then

(i) $\nabla^h_X\overline{Y} - \nabla^h_Y\overline{X} = \overline{[X, Y]}$.
(ii) $X\langle U, V\rangle = \langle\nabla^h_X U, V\rangle + \langle U, \nabla^h_X V\rangle$.

(iii) $\nabla^h_Y \nabla^h_X U - \nabla^h_X \nabla^h_Y U + \nabla^h_{[X,Y]} U = R(\overline{X}, \overline{Y})U$
where $R(\overline{X}, \overline{Y})U(y) := R_{h(y)}\left(\overline{X}(y), \overline{Y}(y)\right)(U(y))$.

To prove e.g. part (i) of Proposition 3.13 one shows that $\nabla^h_X \overline{Y} - \nabla^h_Y \overline{X} - \overline{[X,Y]}$ is $C^\infty(N)$-linear in X and Y and conludes by evaluating on coordinate vector fields. The same works for part (ii) and (iii).

3.2.1. *First Variation Formula.* Let (M, g) be a Riemannian manifold. Recall that for a piecewise C^1-curce $c : [a, b] \to M$, we have defined $l(c) := \int_a^b \|c'(t)\| \, dt$. For many reasons, the *energy* of c, defined by

$$E(c) := \frac{1}{2} \int_a^b \|c'(t)\|^2 \, dt,$$

is a better object to work with due the smoothness and strict convexity of the square of the absolute value, although it depends on the parametrization of c. Note that by Cauchy-Schwarz we have

$$l(c) = \int_a^b 1 \cdot \|c'(t)\| \, dt \leq \left(\int_a^b 1^2 \, dt \right)^{1/2} \left(\int_a^b \|c'(t)\|^2 \, dt \right)^{1/2} = \sqrt{b-a}\sqrt{E(c)}.$$

In other words, $l(c)^2 \leq (b-a)E(c)$ with equality if and only if $\|c'(t)\|$ is constant, i.e. c being parametrized proportional to arc length.

We are interested in the minimal and critical points of thi energy functional.

Lemma 3.14. Let $\gamma : [a, b] \to M$ be a minimizing geodesic connecting p to q. Then for any piecewise C^1-curve $c : [a, b] \to M$ joining p to q we have $E(\gamma) \leq E(c)$ with equality if and only if c is a minimizing geodesic.

Proof. By the above, $(b-a)E(\gamma) = l(\gamma)^2 \leq l(c)^2 \leq (b-a)E(c)$. □

Formalizing the above question, consider the map

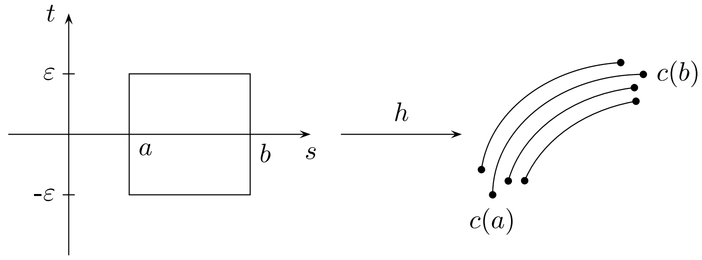
$$E : \Omega_{p,q} := \{c : [a, b] \to M \mid x \text{ is smooth}, \ c(a) = p, \ c(b) = q\} \to \mathbb{R}\,.$$

Then by the above the absolute minima of E arise through the minimal geodesics. We now examime critical points which is made precise by the following.

Definition 3.15. Let M be a manifold and let $c : [a, b] \to M$ be a smooth curve. A *variation of* c is a smooth map

$$h : [a, b] \times (-\varepsilon, \varepsilon) \to M$$

such that $h(s, 0) = c(s)$. For $t \in (-\varepsilon, \varepsilon)$, set $c_t : [a, b] \to M$, $s \mapsto h(s, t)$. The variation h has *fixed endpoints* if $h(a, t) = c(a) = p$ and $h(b, t) = c(b) = q$ for all $t \in (-\varepsilon, \varepsilon)$.



In the context of Definition 3.15, smoothness of h means that it is the restriction of a smooth map defined on an open neighbourhood of $[a,b]\times(-\varepsilon,\varepsilon)$. We let $\partial/\partial t$ and $\partial/\partial s$ denote the coordinate vector fields on $\mathbb{R}^2$.

Lemma 3.16. Let M be a manifold and $c:[a,b]\to M$ be a smooth curve. Further, let $h:[a,b]\times(-\varepsilon,\varepsilon)\to M$ a variation of c. Then $Y(s):=D_{(s,0)}h(\partial/\partial t)$ is a vector field along c. Conversely, given $Y\in\Gamma(c^*\mathrm{T}M)$ there is a variation h of c satisfying $D_{(s,0)}h(\partial/\partial t)$.

Proof. The first assertion is immediate. Conversely, given $Y\in\Gamma(c^*\mathrm{T}M)$ set

$$h(s,t):=\exp_{c(s)}tY(s)$$

which by a compactness argument is defined for $t\in(-\varepsilon,\varepsilon)$ for some $\varepsilon>0$. □

Theorem 3.17. Let (M,g) be a connected Riemannian manifold.

(i) Let $c:[a,b]\to M$ a smooth curve and let h be a variation of c. Define $Y\in\Gamma(c^*\mathrm{T}M)$ by $Y(s):=D_{(s,0)}h(\partial/\partial t)$. Then

$$\left.\frac{d}{dt}\right|_{t=0}E(c_t)=g(Y(s),c'(s))\big|_a^b-\int_a^b g(Y(s),\nabla_{c'(s)}c'(s))\,ds.$$

(ii) The critical points of $E:\Omega_{p,q}\to\mathbb{R}$, i.e. curves c for which

$$\left.\frac{d}{dt}\right|_{t=0}E(c_t)=0$$

for all variations of c with fixed endpoints, are exactly the geodesics connecting p to q.

Proof. Given the definition of the energy functional, we compute $g(c_t'(s),c_t'(s))$ and integrate the result over $s\in[a,b]$. Let N be an open neighbourhood of $[a,b]\times(-\varepsilon,\varepsilon)$ in $\mathbb{R}^2$ so that $h:N\to M$ is smooth. Then

$$\left(\overline{\frac{\partial}{\partial s}}\right)_{(s_0,t_0)}=D_{(s_0,t_0)}h\left(\frac{\partial}{\partial s}\right)=c_{t_0}'(s_0).$$

We therefore obtain using Proposition 3.12 and 3.13:

$$\frac{d}{dt}g(c_t'(s),c_t'(s))=\frac{d}{dt}g\left(\overline{\frac{\partial}{\partial s}},\overline{\frac{\partial}{\partial s}}\right)=2g\left(\nabla^h_{\frac{\partial}{\partial t}}\overline{\frac{\partial}{\partial s}},\overline{\frac{\partial}{\partial s}}\right).$$

Also, by Proposition 3.13 we have

$$\nabla^h_{\frac{\partial}{\partial t}}\overline{\frac{\partial}{\partial s}}-\nabla^h_{\frac{\partial}{\partial s}}\overline{\frac{\partial}{\partial t}}=\overline{\left[\frac{\partial}{\partial t},\frac{\partial}{\partial s}\right]}=0.$$

Hence the above computation continues as

$$=2g\left(\nabla^h_{\frac{\partial}{\partial s}}\overline{\frac{\partial}{\partial t}},\overline{\frac{\partial}{\partial s}}\right)=2\left(\frac{d}{ds}g\left(\overline{\frac{\partial}{\partial t}},\overline{\frac{\partial}{\partial s}}\right)-g\left(\overline{\frac{\partial}{\partial t}},\nabla^h_{\frac{\partial}{\partial s}}\overline{\frac{\partial}{\partial s}}\right)\right)$$

Finally, note that

$$\left.\overline{\frac{\partial}{\partial t}}\right|_{t=0}=D_{(s,0)}h\left(\frac{\partial}{\partial t}\right)=Y(s),\quad \left.\overline{\frac{\partial}{\partial s}}\right|_{t=0}=c'(s),\ \text{and}\ \nabla^h_{\frac{\partial}{\partial s}}\overline{\frac{\partial}{\partial s}}=\nabla_{c'(s)}c'(s).$$

Overall, we therefore obtain

$$\left.\frac{d}{dt}\right|_{t=0}g(c_t'(s),c_t'(s))=2\left(\frac{d}{ds}g(Y(s),c'(s))-g(Y(s),\nabla_{c'(s)}c'(s))\right)$$

which is the assertion after integrating with respect to s.

For the second assertion, note that by (i) we have for every variation of c with fixed endpoints:

$$\frac{d}{dt}\bigg|_{t=0} E(c_t) = -\int_a^b g(Y(s), \nabla_{c'}c'(s))\ ds$$

If c is a geodesic, then $\nabla_{c'}c' = 0$ and hence $\frac{d}{dt}\big|_{t=0} E(c_t) = 0$. Conversely, assume that the above integral vanishes for all variations of c with fixed endpoints. Let $f : [a,b] \to \mathbb{R}$ be a smooth function with $f(a) = 0 = f(b)$ and $f(s) > 0$ for all $s \in (a,b)$. Set $Y(s) := f(s)\nabla_{c'}c'(s)$. Then

$$\frac{d}{dt}\bigg|_{t=0} E(c_t) = -\int_a^b f(s)\|\nabla_{c'}c'(s)\|^2\ ds = 0$$

Then $\nabla_{c'}c'$ vanishes on (a,b) and hence also at a and b by continuity which shows that c is a geodesic. □

3.2.2. *Second Variation Formula.* The second variation formula concerns the second derivative of the energy functional. Retain the above notation and define in addition: $Y(s,t) := D_{(s,t)}h(\partial/\partial t)$. Then Y is a smooth vector field along h.

Theorem 3.18. Retain the above notation. Let $c : [a,b] \to M$ be a geodesic. Then

$$\frac{1}{2}\frac{d^2}{dt^2}\bigg|_{t=0} E(c_t) = g\left(\nabla^h_{\frac{\partial}{\partial t}} Y(s,0), c'(s)\right)\bigg|_a^b + \int_a^b \left\|\nabla^h_{\frac{\partial}{\partial s}} Y(s)\right\|^2 - (Y,c',Y,c')(s,0)\ ds$$

In the context of Theorem 3.18, recall that

$$(Y,c',Y,c')(s,0) = g(R_{c(s)}(Y(s,0),c'(s))Y(s,0),c'(s)).$$

Proof. We determine an expression for

$$\frac{d^2}{dt^2}\bigg|_{t=0} g(c_t'(s), c_t'(s))$$

which we integrate with respect to s in order to obtain the second variation of the energy functional. In view of Proposition 3.13 we have

$$\overline{\frac{\partial}{\partial s}}(s,t) = D_{(s,t)}h\left(\frac{\partial}{\partial s}\right) = c_t'(s) \quad \text{and} \quad \overline{\frac{\partial}{\partial t}}(s,t) = D_{(s,t)}h\left(\frac{\partial}{\partial t}\right) = Y(s,t).$$

To begin with, we compute

$$\frac{d}{dt}g(c_t'(s), c_t'(s)) = \frac{d}{dt}g\left(\overline{\frac{\partial}{\partial s}}, \overline{\frac{\partial}{\partial s}}\right) = 2g\left(\nabla^h_{\frac{\partial}{\partial t}}\overline{\frac{\partial}{\partial s}}, \overline{\frac{\partial}{\partial s}}\right) = 2g\left(\nabla^h_{\frac{\partial}{\partial s}}\overline{\frac{\partial}{\partial t}}, \overline{\frac{\partial}{\partial s}}\right).$$

Therefore we obtain

$$\begin{aligned}\frac{1}{2}\frac{d^2}{dt^2} &= \frac{d}{dt}g\left(\nabla^h_{\frac{\partial}{\partial s}}\overline{\frac{\partial}{\partial t}}, \overline{\frac{\partial}{\partial s}}\right) \\ &= \underbrace{g\left(\nabla^h_{\frac{\partial}{\partial t}}\nabla^h_{\frac{\partial}{\partial s}}\overline{\frac{\partial}{\partial t}}, \overline{\frac{\partial}{\partial s}}\right)}_{(1)} + \underbrace{g\left(\nabla^h_{\frac{\partial}{\partial s}}\overline{\frac{\partial}{\partial s}}, \nabla^h_{\frac{\partial}{\partial t}}\overline{\frac{\partial}{\partial s}}\right)}_{=\left\|\nabla^h_{\frac{\partial}{\partial s}}Y\right\|^2}\end{aligned}$$

The second term in the above sum is in its final term. We continue with the first one using Proposition 3.13

$$(1) = \underbrace{g\left(\nabla^h_{\frac{\partial}{\partial s}}\nabla^h_{\frac{\partial}{\partial t}}\overline{\frac{\partial}{\partial t}}, \overline{\frac{\partial}{\partial s}}\right)}_{(2)} + \underbrace{g\left(R\left(\overline{\frac{\partial}{\partial s}}, \overline{\frac{\partial}{\partial t}}\right)\overline{\frac{\partial}{\partial t}}, \overline{\frac{\partial}{\partial s}}\right)}_{=-(Y,c',Y,c')}$$

Again, the second term is final. We continue with (2):

$$(2) = \frac{d}{ds} g\left(\nabla_{\frac{\partial}{\partial t}} \overline{\frac{\partial}{\partial t}}, \overline{\frac{\partial}{\partial s}}\right) - g\left(\nabla^h_{\frac{\partial}{\partial t}} \overline{\frac{\partial}{\partial t}}, \nabla^h_{\frac{\partial}{\partial s}} \overline{\frac{\partial}{\partial s}}\right).$$

Evaluating at $t = 0$ and using that

$$\nabla^h_{\frac{\partial}{\partial s}} \overline{\frac{\partial}{\partial s}} = \nabla^h_{\frac{\partial}{\partial s}} c'(s) = \nabla_{c'(s)} c'(s) = 0$$

since c is a geodesic we obtain overall:

$$\frac{1}{2} \frac{d^2}{dt^2}\bigg|_{t=0} E(c_t) = \left\|\nabla^h_{\frac{\partial}{\partial s}} Y(s,0)\right\|^2 - (Y, c', Y, c')(s,0) + \frac{d}{ds} g\left(\nabla^h_{\frac{\partial}{\partial t}} Y(s,0), c'(s)\right)$$

which implies the assertion. □

3.3. **Curvature and Topology.** Finally, we are in a position to discuss some interesting applications. In fact, a large amount of recent research in Riemannian Geometry has focused on the relation between the curvature of a Riemannian manifold and its global topological properties. Ricci curvature in particular is involved in many interesting statements. Recall that given a Riemannian manifold (M, g), $p \in M$ and $x \in \mathrm{T}_p M$ we have defined

$$\mathrm{Ric}_p(s) = \frac{1}{(m-1)} \sum_{i=1}^{m-1} \langle R(x, z_i)x, z_i\rangle$$

where $z_1, \ldots, z_{m-1}, z_m = x$ is an orthonormal basis of $\mathrm{T}_p M$. As an example, consider $S^m(r)$, the sphere of radius r. In this case, $\mathrm{Ric}_p(x) = 1/r^2$ for all $p \in S^m(r)$ and $x \in \mathrm{T}_p S^m(r)$. Furthermore, the diamater of $S^m(r)$ is given by $\mathrm{diam} S^m(r) = \pi r$. This example is the extremal case in the following Theorem.

Theorem 3.19 (Bonnet-Myers). Let M be a complete connected Riemannian manifold with $\mathrm{Ric}_p(x) \geq 1/r^2 > 0$ for all $(p, x) \in \mathrm{T}M$. Then M is compact and in fact $\mathrm{diam}(M) \leq \pi r$.

It is a theorem of Schenk that equality in the second inequality fo Theorem 3.19 implies that M is isometriec to a sphere of radius r. This relates to eigenvalues of the Laplacian among other things.

Proof. (Theorem 3.19). By Theorem 2.68 it suffices to show that all minimizing geodesics have length at most πr. Let $p, q \in M$ and let $\gamma : [0,1] \to M$ be a minimizing geodesic with $\gamma(0) = p$ and $\gamma(1) = q$. Let $l = d(p, q)$. Then we may extend $\gamma'(0)/l$ to an orthonormal basis $e_1, \ldots, e_{m-1}, e_m = \gamma'(0)/l$ of $\mathrm{T}_p M$. Let $e_1(s), \ldots, e_{m-1}(s)$ be the parallel vector fields along γ with initial conditions $e_1, \ldots, e_{m-1}$ and define $Y_j(s) = \sin(\pi s) e_j(s)$. Then $Y_j(0) = 0$ and $Y_j(1) = 0$. Finally, let $h_j : [0,1] \times (-\varepsilon, \varepsilon) \to M$ be a variation of c with fixed end points and $D_{(s,0)} h\left(\frac{\partial}{\partial t}\right) = Y_j(s)$. We now compute the second variation of the energy of the map $s \mapsto h_j(s,t)$, denoted by $E''_j(0)$. Since

$$\nabla^h_{\frac{\partial}{\partial s}} Y_j(s,0) = \nabla^h_{\frac{\partial}{\partial s}} (\sin(\pi s) e_j(s)) = \pi \cos(\pi s) e_j(s) + \sin(\pi s) \underbrace{\nabla^h_{\frac{\partial}{\partial s}} e_j(s)}_{=0}$$

we have $\|\nabla^h_{\partial/\partial s} Y_j(s,0)\|^2 = \pi^2 \cos(\pi s)^2$. Furthermore,

$$(Y_j, \gamma', Y_j, \gamma')_{(s,0)} = (\sin \pi s)^2 l^2 \left(e_j(s), e_m(s), e_j(s), e_m(s)\right)(s,0)$$

and hence

$$\frac{1}{2} E''_j(0) = \int_0^1 \pi^2 \cos(\pi s)^2 - l^2 \sin(\pi s)^2 K(e_j(s), e_m(s))\, ds$$

Partial integration of the $\pi^2 \cos(\pi s)^2$ term yields

$$\frac{1}{2}E_j''(0) = \int_0^1 \sin(\pi s)^2 \left(\pi^2 - l^2 K(e_j(s), e_m(s))\right)\, ds$$

and therefore

$$\frac{1}{2(m-1)} \sum_{j=0}^{m-1} = \int_0^1 \sin(\pi s)^2 \left(\pi^2 - l^2 \mathrm{Ric}_{c(s)}(e_m(s))\right)\, ds \leq \int_0^1 \sin(\pi s)^2 \left(\pi^2 - l^2/r^2\right)$$

Now, if $l \gneq \pi r$ then $\sum_{i=1}^{m-1} E_j''(0) \lneq 0$. Hence $E_j''(0) \lneq 0$ for some $j \in \{1, \ldots, m-1\}$ which contradicts γ being minimizing. □

Corollary 3.20. Let M be a complete connected Riemannian manifold with $\mathrm{Ric}_p(x) \geq \delta > 0$ for all $(p, x) \in \mathrm{T}^1 M$. Then the universal covering $\widetilde{M}$ of M is compact and $\pi_1(M)$ is finite.

Proof. Since the covering map is a local isometry, $\widetilde{M}$ satisfies the same curvature bounds as M. Hence $\widetilde{M}$ is compact by Theorem 3.19 and $\pi_1(M)$ is finite. □

As usual, Lie groups with bi-invariant metrics constitute a particularly nice class of examples.

Corollary 3.21. Let G be a connected Lie group which admits a bi-invariant metric and let $\mathfrak{g} := \mathrm{Lie}(G)$. Assume that $\mathrm{Z}(\mathfrak{g}) = 0$. Then both G and $\widetilde{G}$ are compact and $\pi_1(G)$ is finite.

The assumption of Corollary 3.21 that $\mathrm{Z}(\mathfrak{g})$ be trivial is equivalent to the center of G being zero-dimensional.

Proof. (Corollary 3.21). Since G acts on itself by isometric left-translations, we have $\mathrm{Ric}_g(D_e L_g(v)) = \mathrm{Ric}_e(v)$ for all $v \in \mathfrak{g}$ with $|v| = 1$. Hence it suffices to bound $\mathrm{Ric}_e(v)$ from below by a positive quantity. Recall that for orthonormal $X, Y \in \mathfrak{g}$ we have $K(X, Y) = \frac{1}{4}\|[X, Y]\|^2$. Now, let $Y_1, \ldots, Y_{m-1}, Y_m = X$ be an orthonormal basis of $\mathfrak{g}$. Then

$$\mathrm{Ric}_e(X) = \frac{1}{m-1} \sum_{i=1}^{m-1} K(X, Y_i) = \frac{1}{m-1} \sum_{i=1}^{m-1} \|[X, Y]\|^2 \geq 0$$

with equality if and only if $[X, Y_i] = 0$ for all $i \in \{1, \ldots, m-1\}$ which in turn is equivalent to $X \in \mathrm{Z}(\mathfrak{g})$, i.e. $X = 0$ which contridicts $\|X\| = 1$. Therefore, the Ricci curvature being a positive function on a compact sphere, it is bounded below by some δ which implies the assertions by Theorem 3.19. □

As a remark in the context of the proof of Corollary 3.21 we state that

$$\begin{aligned}
\mathrm{Ric}_e(X) &= \frac{1}{m-1} \sum_{i=1}^{m} \frac{1}{4}\|[X, Y_i]\|^2 \\
&= \frac{1}{4(m-1)} \sum_{i=1}^{m} \langle \mathrm{ad}(X) Y_i, \mathrm{ad}(X) Y_i \rangle \\
&= -\frac{1}{4(m-1)} \sum_{i=1}^{m} \langle \mathrm{ad}(X)^2 Y_i, Y_i \rangle \\
&= -\frac{1}{4(m-1)} \operatorname{tr}(\mathrm{ad}(X)^2) \\
&= -\frac{1}{4(m-1)} K_{\mathfrak{g}}(X, X)
\end{aligned}$$

where $K_{\mathfrak{g}}$ denotes the Killing form of $\mathfrak{g}$ defined by

$$K_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R},\ (X, Y) \mapsto \mathrm{tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y))$$

Using this, one obtains more information about the diameter of such Lie groups in terms of root systems.

Corollary 3.22. Let G be a connected compact Lie group with zero-dimensional center. Then $\widetilde{G}$ is compact and $\pi_1(G)$ is finite.

Proof. A compact Lie group always admits a bi-invariant metric. □

Another application of the second variation formula is the following.

Theorem 3.23 (Weinstein). Let M be a compact connected oriented Riemannian manifold which has everywhere strictly positive sectional curvature. Further let $f \in \mathrm{Iso}(M)$ be orientation-preserving if $\dim M$ is even and orientation-reversing if $\dim M$ is odd. Then f has a fixed point.

The example of the two-dimensional sphere and the antipodal map shows that the assumptions on f are necessary.

Before turning to the proof of Theorem 3.23, consider the following linear version.

Lemma 3.24. Let $A \in \mathrm{O}(m-1)$ and suppose that $\det A = (-1)^m$. Then A fixes a non-trivial vector.

Proof. We show that $1 \in \mathbb{R}$ is an eigenvalue of A: Let $\lambda_1, \dots, \lambda_r, \mu_1, \overline{\mu}_1, \dots, \mu_k, \overline{\mu}_k$ be the eigenvalues of A, listed with multiplicity, with $\lambda_i \in \mathbb{R}$ for all $i \in \{1, \dots, r\}$ and $\mu_i \neq \overline{\mu}_i$ for all $i \in \{1, \dots, n\}$. Since A is orthgonal, $\lambda_i \in \{\pm 1\}$ for all $i \in \{1, \dots, r\}$ and all μ_i have unit length. Therefore $\prod_{i=1}^{r} \lambda_i = (-1)^m$. Also we have $r+2k = m-1$. Now, assume that m is even. Then $m-1-2k$ is odd and hence so is r. Hence there is $i \in \{1, \dots, r\}$ such that $\lambda_i = 1$. Conversely, if m is odd, then $-1 = \lambda_1 \cdots \lambda_r$ implies $r \geq 1$. Also, r is even since $r = m-1-2k$. Consequently there is again $i \in \{1, \dots, r\}$ such that $\lambda_i = 1$. □

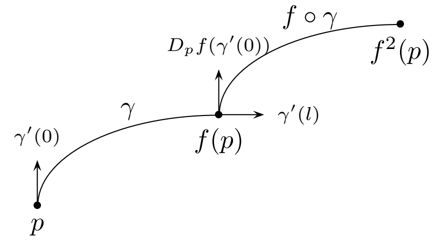
Proof. (Theorem 3.23). We argue by contradiction: Choose $p \in M$ with

$$d(p, f(p)) = \min\{d(q, f(q)) \mid q \in M\}$$

and assume that $0 < d(p, f(p)) =: l$. In this case, let $\gamma : [0, l] \to M$ be a minimizing geodesic connecting p to $f(p)$. First, we claim that $D_p f(\gamma'(0)) = \gamma'(l)$: To this end, consider $0 < t' < l$ and $p' = \gamma(t')$. Then

$$l \leq d(p', f(p')) \leq l([p', f(p)] \cup [f(p), f(p')]) = l.$$

Since the concatenation $[p', f(p)] \cup [f(p), f(p')]$ is a minimizing geodesic and hence C^1 we conclude that $\gamma'(l) = D_p f(\gamma'(0))$.



Now consider $A := P_{\gamma, f(p), p} \circ D_p f \in \mathrm{GL}(\mathrm{T}_p M)$. We have $A(\gamma'(0)) = \gamma'(0)$ as well as $\det A = (-1)^m$. Applying Lemma 3.24 to $A \in \mathrm{O}(\gamma'(0)^\perp)$, there is $e_1 \in \mathrm{T}_p M$, orthogonal to $\gamma'(0)$ which is fixed by A. Let $e_1(s) \in \mathrm{T}_{\gamma(s)} M$ be the vector field parallel to γ with $e_1(0) = e_1$. Then $D_p f(e_1(0)) = e_1(l)$.

Now look at the variation $h(s,t) := \exp_{\gamma(s)}(te_1(s))$. The above implies that $f(h(0,t)) = h(l,t)$, i.e. $f(\gamma_t(0)) = \gamma_t(l)$. Hence

$$l(\gamma_t) \geq d(\gamma_t(0), \gamma_t(l)) = d(\gamma_t(0), f(\gamma_t(0))) \geq l.$$

This implies that $\gamma = \gamma_0$ is a local minimum of $t \mapsto l(\gamma_t)$. Hence $d^2/dt^2|_{t=0} l(\gamma_t) \geq 0$. On the other hand, the second variation formula yields

$$\frac{1}{2} \left.\frac{d^2}{dt^2}\right|_{t=0} E(\gamma_t) = g\left(\nabla_{\frac{\partial}{\partial t}} e_1(s,0), \gamma'(s)\right)\Big|_0^l + \int_0^l \left\|\nabla_{\frac{\partial}{\partial t}} e_1\right\|^2 - (e_1, \gamma', e_1, \gamma')(s,0)\ ds$$
$$= -\int_0^l K(e_1(s), \gamma'(s))\ ds < 0$$

which contradicts the above. □

We now discuss further applications of the second variation formula in both positive and negative curavture.

Corollary 3.25 (Synge). Let M be a compact, connected Riemannian manifold of dimension m with everywhere strictly positive sectional curvature. Then the following hold.

(i) If m is even and M is orientable then M is simply connected.
(ii) If m i sodd then M is orientable.

As a preliminary remark towards the proof of Corollary 3.25 be note that every connected Riemannian manifold M admits an orientable cover. Indeed, set

$$\overline{M} := \{(p, O_p) \mid p \in M,\ O_p \text{ orientation on } \mathrm{T}_pM\}.$$

It is obvious that one can equip $\overline{M}$ with the structure of a smooth manifold such that $p : \overline{M} \to M$, $(p, O_p) \to p$ is a two-sheeted covering which is connected if and only if M is not orientable. It is a Galois covering with non-trivial covering transformation $\sigma : (p, O_p) \mapsto (p, \overline{O}_p)$ where $\overline{O}_p$ is the orientation opposite $\overline{O}_p$.

Proof. (Corollary 3.25). For the first assertion, let $p : \widetilde{M} \to M$ be the universal cover of M and let $\widetilde{g}$ be the Riemannian metric on $\widetilde{M}$ turning p into a Riemannian covering. In particular, if $\delta > 0$ is a lower bound on the sectional curvature of M then δ is also a lower bound on the sectional curvature of $(\widetilde{M}, \widetilde{g})$. Therefore, Theorem 3.19 implies that $\widetilde{M}$ is compact. Now, let $\pi_1(M)$ act by covering transformations on $\widetilde{M}$; they are orientation-preserving isometries since M is orientable. If $\pi_1(M) \neq \{e\}$ then there is $f \in \pi_1(M)\backslash\{\mathrm{Id}\}$ which by Theorem 3.23 has at least one fixed point. This, however, contradicts the fact, that non-trivial covering transformations do not have fixed points.

For part (ii), let m be odd and assume that M is not orientable. In this case, consider the orientable Riemannian double cover $\overline{M}$ of M with Riemannian metric $\overline{g}$ making the covering map Riemannian. Then the generator σ of $\pi_1(M)$ can be viewed as an isometry of $\overline{M}$ and by Theorem 3.23 has a fixed point which yields the same contradiction as before. □

3.3.1. *Jacobi Fields.* In this section we introduce another important, namely Jacobi vector fields, which will allow us to give a description of the exponential map. To motivate this, let's rewrite the integral part of the second variation formula: Since

$$\left\langle \nabla_{\frac{\partial}{\partial s}} Y, \nabla_{\frac{\partial}{\partial s}} Y \right\rangle = \frac{d}{ds}\left\langle Y, \nabla_{\frac{\partial}{\partial s}} Y \right\rangle - \left\langle Y, \nabla^2_{\frac{\partial}{\partial s}} \right\rangle,$$

the integral term reads

$$\int_a^b \|\nabla_{\frac{\partial}{\partial s}} Y\|^2 - (c', Y, c', Y)(s,0)\ ds = \left\langle Y, \nabla_{\frac{\partial}{\partial s}} \right\rangle\Big|_a^b - \int_a^b \left\langle Y, \nabla^2_{\frac{\partial}{\partial s}} Y + R(c', Y)c' \right\rangle\ ds.$$

Therefore,

$$\left.\frac{d^2}{dt^2}\right|_{t=0} E(c_t) = \begin{array}{c}\text{stuff depending on}\\ \text{endpoints only}\end{array} - \int_a^b \left\langle Y, \nabla^2_{\frac{\partial}{\partial s}} Y + R(c', Y)c' \right\rangle \, ds$$

Here, the symmetric bilinear form $-\int_a^b \left\langle Y, \nabla^2_{\frac{\partial}{\partial s}} Y + R(c', Y)c' \right\rangle \, ds$ can be viewed as the second derivative of the energy functional at c and Jacobi vector fields are vector fields in directions where "nothing happens", reflecting degeneracy.

Definition 3.26. Let M be a Riemannian manifold and let c be a geodesic into M. A *Jacobi vector field* is a vector field Y along c satisfying

$$\nabla_{\frac{\partial}{\partial s}} Y + R(c', Y)c' = 0.$$

The following are the main statement we shall prove about Jacobi vector fields.

Theorem 3.27. Let M be a Riemannian manifold and let c be a geodesic into M. Further, let $u, v \in \mathrm{T}_{c(0)}M$. Then there is exactly one Jacobi vector field Y along c with $Y(0) = u$ and $Y'(0) := \nabla_{\partial/\partial s} Y(0) = v$.

Proposition 3.28. Let M be a Riemannian manifold, $c : [a, b] \to M$ a geodesic and $h : [a, b] \times (-\varepsilon, \varepsilon) \to M$ a variation of c such that for every $t \in (-\varepsilon, \varepsilon)$, the curve $c_t : [a, b] \to M$ is a geodesic. Then $Y(s) := D_{(s,0)}h(\partial/\partial t)$ is a Jacobi vector field along c. Coversely, every Jacobi vector field can be obtained in this way.

Proof. (Theorem 3.27). Fix an orthonormal frame $X_1, \ldots, X_m$ of parallel vector fields along c using that parallel transport is isometric. Then any vector field Y along c can be expressed as

$$Y(s) = \sum_{i=1}^{m} y_i(s) X_i(s)$$

for some smooth functions y_i. Since the X_i are parallel along c we conclude

$$\nabla^2_{\frac{\partial}{\partial s}} Y(s) = \sum_{i=1}^{m} y_i''(s) X_i(s).$$

Therefore,

$$\nabla^2_{\frac{\partial}{\partial s}} Y + R(c', Y)c' = 0 \;\Leftrightarrow\; \left\langle \nabla^2_{\frac{\partial}{\partial s}} Y + R(c', Y)c', X_i \right\rangle = 0 \quad \forall 1 \le i \le m.$$

Replacing Y by its expression in the X_i leads to the following system of ordinary differential equations:

$$y_i''(s) + \sum_{j=1}^{m} y_j(s) \langle c', X_j, c', X_i \rangle (s) = 0.$$

Standard results now imply the theorem. □

We now turn to Proposition 3.28.

Proof. (Proposition 3.28). For the first part, assume that $h : [a, b] \times (-\varepsilon, \varepsilon)$ is a variation of c such that all c_t are geodesics and compute

$$\nabla^2_{\frac{\partial}{\partial s}} Y = \nabla_{\frac{\partial}{\partial s}} \nabla_{\frac{\partial}{\partial s}} \overline{\frac{\partial}{\partial t}} = \nabla_{\frac{\partial}{\partial s}} \nabla_{\frac{\partial}{\partial t}} \overline{\frac{\partial}{\partial s}} = \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial s}} \overline{\frac{\partial}{\partial s}} + R\left(\overline{\frac{\partial}{\partial t}}, \overline{\frac{\partial}{\partial s}}\right) \overline{\frac{\partial}{\partial s}}.$$

Now note that

$$\overline{\frac{\partial}{\partial s}}(s, t) = D_{(s,t)} h \frac{\partial}{\partial s} = c_t'(s).$$

Because of this and the fact that all c_t are geodesics we obtain evaluating at $t = 0$:

$$\nabla^2_{\frac{\partial}{\partial s}} Y = R(Y, c')c' = -R(c', Y)c'.$$

For the converse, let Y be a Jacobi vector field along c and let $\gamma : (-\varepsilon, \varepsilon) \to M$ be a geodesic with $\gamma(0) = c(0)$. Also, let X be a vector field along γ. Both are going to be determined later. In any case, $h(s,t) := \exp_{\gamma(t)}(sX(t))$ has the property that for each t the map $s \mapsto h(s,t)$ is a geodesic. We have $h(s,0) = \exp_{c(0)}(sX(0))$. We are going to choose $X(t) = X_1(t) + tX_2(t)$ for some parallel vector fields X_1, X_2 along γ with $X_1(0) = c'(0)$. For the moment, compute the Jacobi vector field associated to this geodesic variation.

$$\overline{\frac{\partial}{\partial t}} = D_{(s,0)} h \frac{\partial}{\partial t}.$$

Unable to do this directly, we look at the initial conditions

$$\overline{\frac{\partial}{\partial t}}(0,0) \quad \text{and} \quad \nabla_{\frac{\partial}{\partial s}} \overline{\frac{\partial}{\partial t}}(0,0).$$

Regarding the first, we have

$$\overline{\frac{\partial}{\partial t}}(0,0) = D_{(0,0)} h \frac{\partial}{\partial t} = \left.\frac{d}{dt}\right|_{t=0} (t \mapsto \exp_{\gamma(t)}(0) = \gamma(t)) = \gamma'(0)$$

Therefore, we set, $\gamma'(0) = Y(0)$. Regarding the second, we have

$$\nabla_{\frac{\partial}{\partial s}} \overline{\frac{\partial}{\partial t}}(s=0) = \nabla_{\frac{\partial}{\partial t}} \overline{\frac{\partial}{\partial s}}(s=0).$$

Using that

$$\left.\overline{\frac{\partial}{\partial s}}\right|_{s=0} = D_{(0,t)} h \frac{\partial}{\partial s} = X(t) = X_1(t) + tX_2(t)$$

we conclude

$$\nabla_{\frac{\partial}{\partial t}} \overline{\frac{\partial}{\partial s}}(0,t) = X_2(t)$$

and hence choose $X_2(0) = Y'(0)$. □

The following result finally links Jacobi vector fields to the derivative of the expoential map.

Proposition 3.29. Retain the above notation. Let $u, v \in \mathrm{T}_p M$ and $c(s) := \exp(su)$. Further, let Y be the Jacobi vector field along c with $Y(0) = 0$ and $Y'(0) = 0$. Then $D_{sv}(\exp_p)(su) = Y(s)$.

Proof. Consider the variation of c given by $h(s,t) := \exp_p(s(v+tu))$. Then we have $h(s,0) = \exp(sv) = c(s)$ and $h(0,t) = p = c(0) = c_t(0)$. We compute the Jacobi vector field Z associated to h. In fact, in view of Theorem 3.27, we determine $Z(0)$ and $Z'(0)$. Obviously, $Z(0) = 0$. Next,

$$Z'(0) = \nabla_{\frac{\partial}{\partial s}} \overline{\frac{\partial}{\partial t}}(0,0) = \nabla_{\frac{\partial}{\partial t}} \overline{\frac{\partial}{\partial s}}(0,0)$$

where

$$\overline{\frac{\partial}{\partial s}} = D_{(s,t)} h \frac{\partial}{\partial s} = D_{s(v+tu)} \exp_p(v+tu)$$

At $s = 0$ we therefore have $\overline{\partial/\partial s}(0,t) = v + tu$. Hence $Z'(0) = u$. Therefore, $Z(s) = Y(s)$ but by definition

$$Z(s) = \left.\frac{\partial}{\partial t}\right|_{t=0} \exp_p(s(v+tu)) = D_{su} \exp_p(su).$$

□

As an application of the theory of Jacobi fields, we present the following arguments to determine the sectional curvature of spheres and hyperbolic space, bypassing several pages of computations.

Consider the sphere S^n. Let $x \in S^n$ and pick $v \in S^n$ with $\langle x, v\rangle = 0$. Then

$$c(s) := (\cos s)x + (\sin s)v$$

is a geodesic. Now pick $u \in S^n$ with $\langle u, x\rangle = 0 = \langle u, v\rangle$ and consider the variation

$$h(s,t) := \cos s + \sin s((\cos t)v + (\sin t)u).$$

Then $h(s, 0) = c(s)$ and $h(0,t) = x$. Also,

$$Y(s) := \overline{\frac{\partial}{\partial t}} = (sins)U(s)$$

for a vector field U along c with $U(s) = u$. Then

$$\nabla_{\frac{\partial}{\partial s}} Y = (\cos s)U(s) \quad \text{and} \quad \nabla^2_{\frac{\partial}{\partial s}} Y = -(\sin s)U(s)$$

because U is parallel along c given that parallel transport is an orientation-preserving isometry. We thus have

$$0 = \langle \nabla^2_{\frac{\partial}{\partial s}} Y + R(c', Y)c', Y\rangle \quad \Leftrightarrow \quad 0 = -\sin^2 s + \sin^2 s(c'u, c', u)$$

for all s with $\sin(s) \neq 0$, implying $(c', u, c', u) = 1$ for such s and by continuity $(u, v, u, v) = 1$.

The case of hyperbolic space is left as an exercise.

3.3.2. *The Cartan-Hadamard Theorem.* In this section, we see an example of how negative curvature impacts on the topology of manifolds.

Theorem 3.30 (Cartan-Hadamard). Let (M, g) be a complete Riemannian manifold with everywhere non-positive sectional curvature. Then $\exp_p : \mathrm{T}_pM \to M$ is a covering map.

In the context of Theorem 3.30 note that if M is simply connected then $\exp_p$ is a diffeomorphism. Its proof relies on the following lemma.

Lemma 3.31. Let (M, g) be a complete Riemannian manifold with everywhere non-positive sectional curvature. Then $\exp_p : \mathrm{T}_pM \to M$ is a local diffeomorphism.

Proof. Here, we utilize our knowledge of the derivative of the exponential map as well as the inverse function theorem. Using the notation of Proposition 3.29, let $Y(s) = D_{sv} \exp_p(su)$ and assume $u \neq 0$. We know that $Y(s) = 0$ and $Y'(0) = u$. Now consider the function $f(s) := \langle Y(s), Y(s)\rangle$. Then

$$f'(s) = 2\left\langle \nabla_{\frac{\partial}{\partial s}} Y(s), Y(s)\right\rangle$$

and

$$''(s) = 2\left\langle \nabla_{\frac{\partial}{\partial s}} Y(s), \nabla_{\frac{\partial}{\partial s}} Y(s)\right\rangle + 2\left\langle \nabla^2_{\frac{\partial}{\partial s}} Y(s), Y(s)\right\rangle.$$

where

$$\left\langle \nabla^2_{\frac{\partial}{\partial s}} Y, Y\right\rangle = -(c', Y, c', Y) \geq 0$$

which implies $f''(s) \geq 0$. Also, we have $f(0) = 0$ and $f'(0) = 0$. From the first, we conclude that f' is increasing. In particular, since $f'(0) = 0$ we get $f'(s) \geq 0$ for all $s \in [0, \infty)$. Hence f is increasing and positive. Assume there is $s_0 > 0$ with $f(s_0) = 0$ which implies $f(s) = 0$ for all $s \in [0, s_0]$. Then $f''(0) = 0$ for all $s \in (0, s_0)$, i.e.

$$\|\nabla_{\frac{\partial}{\partial s}} Y(s)\| = 0 \ \forall s \in (0, s_0).$$

and by continuity

$$\|\nabla_{\frac{\partial}{\partial s}} Y(0)\| = 0.$$

That is, $u = 0$, a contradiction. □

In order to pass from a local diffeomorphism to a covering map, we shall use the following lemma whose proof is left as an exercise.

Lemma 3.32. Let (N_1, g_1) and (N_2, g_2) be connected Riemannian manifolds and let $p : N_1 \to N_2$ be a local isometry. Assume that (N_1, g_1) is complete. Then p is covering map.

Proof. (Theorem 3.30). Let $\widetilde{g}$ be the pullback of g via $\exp_p$, so that $\exp_p$ is a local isometry. The metric $\widetilde{g}$ is complete because straight lines in T_pM issuing from $0 \in \mathrm{T}_pM$ are geodesics by the Lemma 2.62. □

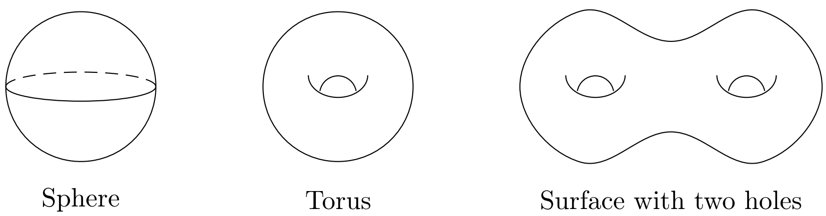
4. What's Beyond

First, we record the following important theorem, see [Wol11].

Theorem 4.1. Let M be a simply connected, complete Riemannian manifold of constant sectional curvature k. Then M is, up to rescaling of the metric, isometric to either $\mathbb{S}^m$, $\mathbb{E}^m$ or $\mathbb{H}^m$.

What about classifying Riemannian manifold of constant sectional curvature in general? To avoid pathologies like taking the complement of a closed set in one of the above, one should require completeness at least. The positive curvature case then amounts to determine, up to conjugacy, finite subgroups G of $\mathrm{O}(m+1)$ such that for every $g \in G\backslash\{\mathrm{Id}\}$, the number one is not an eigenvalue. This is fairly complicated and was done accomplished by Vincent, a student of de Rham in the 1940's. In the flat case, this is still an open question. For instance, Bieberbach groups have only been classified for $m \leq 6$, the case $m = 3$ being particularly classical and used by chemist's ever since. There is a mechanism due to Calabi by which one can understand n-dimensional flat manifolds with $\beta_1(M) > 0$ in terms of lower dimensional ones N with $\beta_1(N) = 0$.

In negative curvature, the story has a completely different flavour. For instance, recall the classification of compact orientable manifolds.

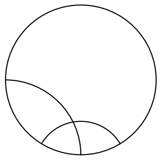


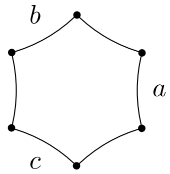
There is the sphere, the torus, and surfaces S_g $(g \geq 2)$ with g holes. The Gauss-Bonnet theorem states that for a compact orientad surface S with sectional curvature $k : M \to \mathbb{R}$ we have

$$\frac{1}{2\pi}\int_S k_g \, d\omega = \chi(S) = 2 - 2g$$

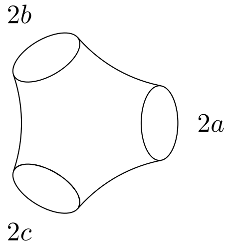
where $\chi(S)$ is the Euler characteristic of S. In particular, only the sphere carries a positively curved metric, only the torus carries a flat metric and only the higher genus surfaces potentially carry hyperbolic metric. In this case, the Gauss-Bonnet Theorem implies that $\mathrm{area}(S) = 4\pi(g-1)$.

In fact, there are many hyperbolic metrics on higher genus surfaces. To describe these one can either turn to the study of discrete subgroups of the isomorphism group of two-dimensional hyperbolic space or use the following explicit approach due to Buser in 1978 involving right-angled hexagons in two-dimensional hyperbolic space. To describe these, we use the Poincaré disk model of the hyperbolic plane which has the advantages that the Euclidean angles one sees are the same as the actual (hyperbolic) angles since at every point in the disk, the hyperbolic metric is a multiple of the Euclidean metric.

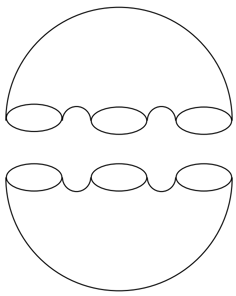




In a right-angled hexagons, bounded by geodesic rays, the side lengths a, b and c are free parameters. Let $H(a, b, c)$ denote the associated hexagon and $\overline{H}(a, b, c)$ the one with opposite orientation. Glueing $H(a, b, c)$ to $\overline{H}(a, b, c)$ yields a pair of pants which closes up nicely due to the fact that the hexagons are right-angled.



A surface of genus two now arises through gluing two pairs of pants along the geodesic boundaries. Here, three more rotational parameters are introduced.



In total, there are six parameters. In the case of arbitrary higher genus g, there are $6g - 6$ parameters.

In higher dimensions, the story yet takes a completely different flavour.

Theorem 4.2 (Mostow). Let M_1 and M_2 be compact Riemannian manifolds of constant sectional curvature $k = -1$ and of dimension at least three. Then any isomorphism of $\pi_1(M_1)$ and $\pi_1(M_2)$ is induced by an isometry of M_1 and M_2.

In particular, such a manifold carries only one hyperbolic metric. Understanding the isomorphism classes of such $\pi_1(M)$ inside $\mathrm{SO}(n, 1)$ remains a challenge though.

We also did not touch at all pinching theorems in the spirit of Berger-Klingenberg mentioned in the introduction.

A revolutionary type of result was introduced by Gromov. The basic idea dates back to Ulam who introduced the study of small perturbations of notions in algebra. For instance, instead of studying the homomorphism equation $\varphi(ab) - \varphi(a) - \varphi(b) = 0$, one may, in the presence of a metric, ask for the left-hand-side to be bounded and see what remains. In a Riemannian geometric setting, one may want to study manifolds for which the absolute value of the sectional curvature is bounded. Or rather, since the metric can always be scaled, ask for the product of the absolute value of the sectional curvature and the diameter of the manifold to be bounded. If the manifold has zero curvature and is compact then its fundamental group contains $\mathbb{Z}^m$ and is virtually abelian. If one only asks for a small bound, nilpotent fundamental groups arise and it is an influential theorem of Gromov stating that this is always the case in a certain sense.

Theorem 4.3. For every $n \geq 1$ there is a constant $\varepsilon_n > 0$ such that if (M, g) is a compact Riemannian n-manifold with

$$|k|\mathrm{diam}(M)^2 < \varepsilon_n$$

then there is a finite covering of M of the form $\Gamma\backslash N$ where N is a simply connected nilpotent group and Γ is a lattice. In particular, $\pi_1(M)$ is virtually nilpotent.

For instance, the manifold

$$N := \left\{ \begin{pmatrix} 1 & x & y \\ & 1 & z \\ & & 1 \end{pmatrix} \middle| \, x, y, z \in \mathbb{R} \right\}$$

admits such an almost flat metric and the quotient $N(\mathbb{Z})\backslash N$ has nilpotent fundamental group $N(\mathbb{Z})$.

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