DIFFERENTIAL GEOMETRY

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Abstract. These are notes of the course Differential Geometry I held at ETH Zurich in 2015.

Disclaimer. This is a preliminary version. Please report any typos, mistakes, comments etc. to stephan.tornier@math.ethz.ch.

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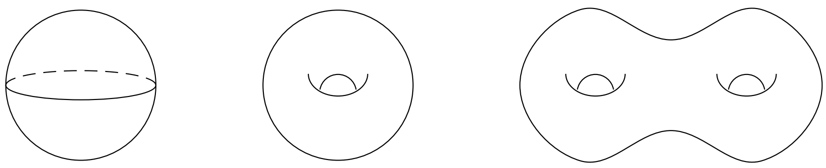
Contents

Introduction

Differential geometry is a synthesis of three different subjects: Analysis in $\mathbb{R}^n$, topology and multilinear algebra. It precisely defines a class of "spaces" on which one can do analysis, termed *differential manifolds*. These spaces as well as the associated notion of *differentiable functions* are the central concept of this course. Differential manifolds look locally like $\mathbb{R}^n$ but are globally much less boring. Examples are the sphere as well as surfaces with holes:



Sphere   Torus   Surface with two holes

An example of what we mean by "do analysis" is the following: If D is a region in $\mathbb{R}^2$ contoured by a curve $\sigma : [0,1] \to \mathbb{R}^2$ and $L, M : \mathbb{R}^2 \to \mathbb{R}$ are reasonably smooth functions, then by Green's formula:

$$\oint_c L(x,y)\ dx + M(x,y)\ dy = \iint_D \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y} \right)\ dx\ dy.$$

Note that whereas the left-hand side of the equation only takes into account the values of L and M on c it yet remembers something about D, namely the right-hand side. Green's formula, the divergence theorem and other well-known formulas are

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all incarnations of Stokes' Theorem which is best understood in the framework of manifolds. In this context we will have to make precise what object makes sense to be integrated over a manifold: *differential forms*. Stokes' Theorem will also be used to define invariants that can e.g. tell the above surfaces apart in the sense that one cannot be deformed into the other without tearing it apart.

Differential geometry forms a basis for many other subjects. Clearly Riemannian geometry is one of them. However, so are Lie groups and physics. The relationship between differential geometry in algebraic geometry is a special one. In a way, notions of one of the two fields echo in the other. For instance, highly abstract algebraic geometric concepts are often easier to visualize in differential geometry.

References for this course include Boothby's [Boo03] which is readable by students, Barden's and Thomas' [BT03] which will in particular be used for differential forms and Milnor's classic [Mil97] on which our section on Brouwer's fixed theorem will be based.

1. Differential Manifolds and Differentiable Maps

1.1. **Differential Manifolds: Definitions and Examples.**
Differential manifolds were first studied by Riemann in 1854 and later on by Poincaré. However, they were still thinking about manifolds being imbedded in some euclidean space and lacked a precise definition. Nevertheless, Stokes' theorem and notions like curvature were already around. The first precise definition, however, was given in 1913 by Weyl at ETH, see [RW13].

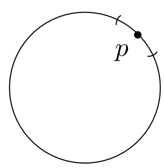
1.2. **Differential Manifolds: Definitions and Examples.**
The first step towards defining differential manifolds is to introduce topological manifolds.

Definition 1.1. An n-dimensional topological manifold is a topological space which is
- (i) Hausdorff (T_2),
- (ii) second countable, and
- (iii) locally homeomorphic to $\mathbb{R}^n$.

Clearly, the last condition of 1.1 is the crucial one. Recall that a topological space X is *Hausdorff* if for every pair of points $x, y \in X$ with $x \neq y$ there are open sets $U, V \subseteq X$ containing x and y respectively such that $U \cap V = \emptyset$. Moreover, X is *second countable* if it its topology admits a countable basis $\mathcal{B} = \{U_n \mid n \in \mathbb{N}\}$, i.e. any open set $U \subseteq X$ can be written as $U = \bigcup_{U_n \subseteq U} U_n$. Eventually, X is *locally homeomorphic to* $\mathbb{R}^n$ if every point in M admits an open neighbourhood which is homeomorphic to an open subset of $\mathbb{R}^n$.

We are going to discuss many examples and non-examples of differential manifolds. Therefore our list of examples and non-examples of topological manifolds is rather short.

Example 1.2.
- (i) Let M be a countable discrete space. Then M is a zero-dimensional topological manifold. The converse is true as well. However, the classification of n-dimensional manifolds is much harder in larger n.
- (ii) The circle $S^1 \subseteq \mathbb{R}^2$ with the induced topology is clearly a topological manifold of dimension one. Every point p admits a neighbourhood that is homeomorphic to an interval:



(iii) The real line $\mathbb{R}$ is a one-dimensional topological manifold as well. This is tautological.

If one restricts oneself to connected, one-dimensional topological manifolds then S^1 and $\mathbb{R}$ are in fact the only examples up to homeomorphism.

(iv) In dimension two, the situation is already so rich that it defies any reasonable classification. Examples are, as before, the family of surfaces with g holes ($g \in \mathbb{N}_0$): These surfaces are all compact and connected and in fact a classification of compact, connected, two-dimensional topological manifolds is managable. To this end we will later on introduce the notion of orientability to distinguish between orientable examples as above and non-orientable examples like projective space. To get an idea of the wealth of general (connected) two-dimensional topological manifolds, note that the complement in $\mathbb{R}^2$ of a Cantor set is an example.

Non-Example 1.3.

(i) Consider $M := [0, 1] \subset \mathbb{R}$ with the induced topology. Every interior point of M satisfies the third requirement of the definition of a topological manifold for $n = 1$ but the points $0, 1 \in M$ do not. For instance, a typical neighbourhood of $0 \in M$ is given by $U = [0, \varepsilon)$ for some $0 < \varepsilon < 1$. Suppose $\varphi : U \to V \subseteq \mathbb{R}$ is a homeomorphism onto an open subset V of $\mathbb{R}$. Note $U\backslash\{0\}$ is connected, but $V\backslash\{\varphi(0)\}$ is not. This contradicts φ being continuous.

(ii) The set $M := [0, 1]^2$ is not a (two-dimensional) topological manifold either. Here one may argue using the fundamental group instead of connectedness.

Both non-examples above are *manifolds with boundary* though, as defined later.

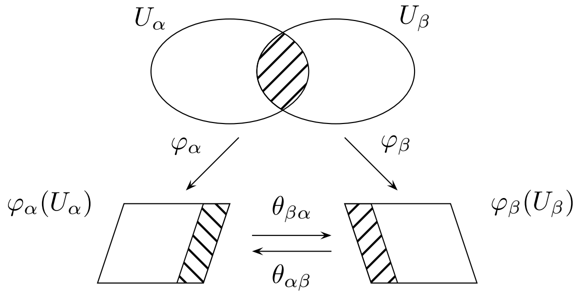
(iii) Let $M = \mathbb{Q} \subset \mathbb{R}$ with the induced topology. Then every open set of $\mathbb{Q}$ is countable and hence cannot be in bijection with an open subset of any $\mathbb{R}^n$.

Remark 1.4. In order to show that the dimension of a topological manifold is well-defined one has to show that if $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ are non-empty and homeomorphic then $n = m$. If $n = 1$ or $m = 1$ one may argue in the above fashion using connectedness. The other cases require e.g. some basic homology theory. If, however, one asks for U and V to be C^1-diffeomorphic then basic linear algebra applied to the derivative readily implies $n = m$. This argument is going to apply in the context of differential manifolds.

The next definition constitutes the next important step towards the definition of differential manifolds.

Definition 1.5. Let M be an n-dimensional topological manifold. A *chart* on M is a pair (U, φ) consisting of an open subset $U \subseteq M$ and a homoeomorphism $\varphi : U \to \varphi(U) =: V \subseteq \mathbb{R}^n$. The subset U is the *coordinate neighbourhood* and V is the *coordinate space*.

We are now going to examine the case in which two charts intersect, see below.



The maps

$$\theta_{\beta\alpha} := \varphi_\beta \circ (\varphi_\alpha|_{U_\alpha \cap U_\beta})^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta),$$

$$\theta_{\alpha\beta} := \varphi_\alpha \circ (\varphi_\beta|_{U_\beta \cap U_\alpha})^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta),$$

are *coordinate transformations* or *change of charts*. They are homeomorphisms between open subsets of $\mathbb{R}^n$ as $\theta_{\beta\alpha}^{-1} = \theta_{\alpha\beta}$.

Definition 1.6. Let M be a topological manifold. A C^0*-atlas* $\mathcal{A}$ on M is a collection of charts $\mathcal{A} = \{(U_\alpha, \varphi_\alpha) \text{ chart on M} \mid \alpha \in A\}$ such that $\bigcup_{\alpha \in A} U_\alpha = M$.

Note that our definition perfectly resembles real-life atlases, introduced by Mercator in 1585. By Definition 1.1, every topological manifold admits a C^0-atlas. As a next step towards the definition of differential manifolds we now introduce smooth atlases.

Definition 1.7. Let $\mathcal{A}$ be a C^0-atlas. Then $\mathcal{A}$ is a C^k*-atlas* if all coordinate transformations between members of $\mathcal{A}$ are C^k-maps. A C^∞-atlas is *smooth*.

Recall that a map $f : U \to \mathbb{R}^m$ from an open subset $U \subseteq \mathbb{R}^n$ to $\mathbb{R}^m$ given by $x \mapsto (f_1(x), \dots, f_m(x))$ where $f_i := \pi_i \circ f$ for all $i \in \{1, \dots, m\}$ is C^k if all partial derivatives of f_i ($i \in \{1, \dots, m\}$) up to order k exist and are continuous. Also, note that in a C^k-atlas all coordinate transformations are in fact C^k-diffeomorphisms by the above, in contrast to the fact that there is a smooth homeomorphism $f : \mathbb{R} \to \mathbb{R}$ which is not a diffeomorphism, for instance $f : x \mapsto x^3$. In particular, this map cannot occur as a coordinate transformation in any smooth atlas.

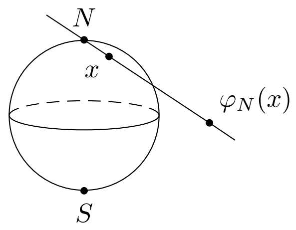
Example 1.8. We now give examples of topological manifolds with smooth atlases.

(i) Let $M := U$ be an open subset of $\mathbb{R}^n$ and $\mathcal{A} = \{(U, \mathrm{id})\}$. This example may seem rather trivial but nevertheless includes an important example of a Lie group, namely $\mathrm{GL}(n, \mathbb{R}) := \{A \in M_{n,n}(\mathbb{R}) \mid \det A \neq 0\} \subseteq M_{n,n}(\mathbb{R}) \cong \mathbb{R}^{n^2}$.

(ii) Let $M := \mathbb{S}^n \subseteq \mathbb{R}^{n+1}$ with the induced topology. If $n \geq 1$ then an atlas of M requires at least two charts since M is compact whereas an open subset of $\mathbb{R}^n$ is not. Now, there are in fact many choices for an atlas with two elements one of which may be constructed using stereographic projection as follows: Let $S^n = \{x = (x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} x_i^2 = 1\}$, $N := (0, \dots, 0, 1) \in S^n$ and $S := (0, \dots, 0, -1)$. Further, set $U_N := S^n \backslash \{N\}$ and $U_S := S^n \backslash \{S\}$. Then the pairs (U_N, φ_N) and (U_S, φ_S), where

$$\varphi_{N,S} : S^n \to \mathbb{R}^n, \ x \mapsto \frac{(x_1, \dots, x_n)}{1 \mp x_{n+1}}$$

are given by stereographic projection, are charts for M.



One checks that the associated coordinate transformation is given by

$$\theta_{SN} : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\},\ y \mapsto \frac{y}{\|y\|^2}.$$

Observe that θ_{SN} exchanges the inside and the outside of the unit sphere in $\mathbb{R}^n$. For $n = 2$ this is an example of a Möbius transformation.

Now, the first attempt to define a smooth manifold would be to say that a smooth manifold is a topological manifold together with a smooth atlas. In this case however, there would be as many smooth manifolds associated to e.g. the sphere as there are atlases on it, which is impractical. We therefore introcoduce the following definitions to talk about *maximal* atlases only.

Definition 1.9. Let M be a topological manifold. Two charts $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) on M are *C^k-compatible* if both coordinate transformations are C^k. A C^k-atlas $\mathcal{A}$ on M is *maximal* if each chart which is compatible with every chart in $\mathcal{A}$ is in $\mathcal{A}$.

Retain the notation of Definition 1.9. If $(U, \varphi) \in \mathcal{A}$ and $\mathcal{A}$ is maximal then every $(V, \varphi|_V)$ where $V \subseteq U$ is open is contained in $\mathcal{A}$ as well. The following lemma is going to come in very handy when defining manifolds. Its proof is left as an exercise.

Lemma 1.10. Let M be a topological manifold. Every C^k-atlas $\mathcal{A}$ on M is contained in a unique maximal atlas.

Sketch of Proof. If a maximal atlas $\mathcal{M}$ containing $\mathcal{A}$ exists it contains the charts of

$$\mathcal{S} := \{(U, \varphi) \mid (U, \varphi) \text{ is a chart on } M \text{ compatible with every chart in } \mathcal{A}\} \supseteq \mathcal{A}\,.$$

Conversely, if the above set is an atlas it is maximal among those containing $\mathcal{A}$ and necessarily unique with this property since any other maximal atlas containing $\mathcal{A}$ consists of charts which are (in particular) compatible with all the charts of $\mathcal{A}$ and is thus contained in $\mathcal{S}$. By maximality it then equals $\mathcal{S}$. Therefore it only remains to show that $\mathcal{S}$ is an atlas. Since $\mathcal{A} \subseteq \mathcal{S}$ and $\mathcal{A}$ is an atlas of M we have

$$M = \bigcup_{(U,\varphi) \in \mathcal{S}} U.$$

As to coordinate transformations, let $(U_\alpha, \varphi_\alpha), (U_\beta, \varphi_\beta) \in \mathcal{S}$. We show that the coordinate transformation $\theta_{\beta\alpha} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ is C^k on a neighbourhood of every point $\varphi_\alpha(x)$ ($x \in U_\alpha \cap U_\beta$) of its domain. Since $\mathcal{A}$ is an atlas there is a chart $(V, \psi) \in \mathcal{A}$ with $x \in V$. Since further both $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) are compatible with (V, ψ) the coordinate transformations $\theta_{\beta\psi}$ and $\theta_{\psi\alpha}$ are smooth and hence so is $\theta_{\beta\alpha}$ at $\varphi_\alpha(x)$ which agrees with $\theta_{\beta\psi} \circ \theta_{\psi\alpha}$ on $U_\alpha \cap U_\beta \cap V \ni x$.

Finally, we are in a position to define the central concept of this course.

Definition 1.11. A *C^k-differential manifold* is a topological manifold with a maximal C^k-atlas. A C^∞-differential manifold is *smooth*.

We shall only concern ourselves with smooth manifolds in this course and not discuss subtleties arising from the exact value of k. One may wonder whether any topological manifold admits a smooth atlas. The following remark answers this question in the negative.

Remark 1.12. There is a compact topological manifold that does not admit a smooth atlas. However it does admit a system of pairwise compatible charts that cover all but one point. See [Ker60].

As to uniqueness of maximal atlases, we remark the following.

Remark 1.13. Let M be a smooth manifold with maximal atlas $\mathcal{A}$. If $F : M \to M$ is a homeomorphism then $\mathcal{A}' := \{(F(U), \varphi \circ F^{-1}) \mid (U, \varphi) \in \mathcal{A}\}$ is a maximal smooth atlas as well. This is readily checked by examining the coordinate transformations. They do not notice the changed point of view at all!

Even more remarkable is the fact that there are topological manifolds which admit incompatible maximal atlases that do not even arise from each other in the above fashion.

Remark 1.14. The topological manifold S^7 admits a maximal atlas which is not compatible with the one defined in Example 1.8 and can even be given by polynomial equations. See [Mil56].

Example 1.15. We now describe three standard ways of producing new manifolds out of old ones.

(i) (Regular value) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a smooth map and let $a \in \mathbb{R}^m$ be a regular value, i.e. for every $x \in f^{-1}(a)$ the derivative $D_x f : \mathbb{R}^n \to \mathbb{R}^m$ has maximal rank m. Then $f^{-1}(a) = \{x \in \mathbb{R}^n \mid f(x) = a\}$ is a smooth manifold in a natural way. Note that all manifolds produced in this way are naturally subsets of $\mathbb{R}^n$. Later on we give the details of this construction in much greater generality.

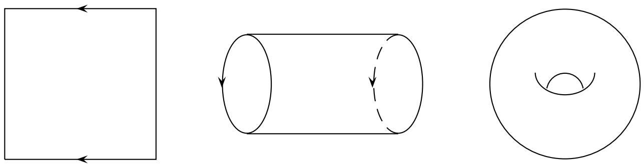
(ii) (Open subset) Let M be a smooth manifold with atlas $\mathcal{A}$ and let $U \subseteq M$ be open. Then $\mathcal{A}|_U := \{(U \cap V, \varphi_{U\cap V}) \mid (V, \varphi) \in \mathcal{A}\}$ is a smooth atlas on U. Note that $\mathcal{A}|_U$ need not be maximal. However, it is contained in a unique maximal smooth atlas by Lemma 1.10.

(iii) (Products) Let M, N be smooth manifolds with atlases $\mathcal{A} = \{(U_\alpha, \varphi_\alpha \mid \alpha \in A\}$ and $\mathcal{B} = \{(V_\beta, \psi_\beta) \mid \beta \in B\}$. Then $\mathcal{P} := \{(U_\alpha \times V_\beta, \varphi_\alpha \times \psi_\beta \mid (\alpha, \beta) \in A \times B\}$ is a smooth atlas on $M \times N$. We recall the definition of $\varphi_\alpha \times \psi_\beta$ ($(\alpha, \beta) \in A \times B$): Let $\varphi_\alpha : U_\alpha \to \mathbb{R}^m$ and $\psi_\beta : V_\beta \to \mathbb{R}^n$ where $m = \dim M$ and $n = \dim N$. Then

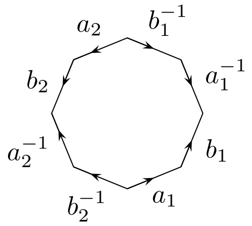
$$\varphi_\alpha \times \psi_\beta : U_\alpha \times V_\beta \to \mathbb{R}^m \times \mathbb{R}^n,\ (x, y) \mapsto (\varphi_\alpha(x), \psi_\beta(x))$$

and $\mathbb{R}^m \times \mathbb{R}^n$ is identified with $\mathbb{R}^{m+n}$. Note that $\mathcal{P}$ is rarely going to be maximal as $U_\alpha \cap V_\beta$ typically has many open subsets that do not have a product structure. Again, Lemma 1.10 turns $\mathcal{P}$ into a maximal atlas.

Next, we describe a construction which does not yield smooth manifolds that are naturally subsets of some euclidean space, namely quotients. As a matter of fact, the construction need not even begin with a smooth manifold. For instance consider a square and glue two opposite sides together. The result is a cylinder. Continuing by gluing together its ends yields a torus which is a smooth manifold.



The reader is invited to think about what the following identification produces. For $i \in \{1, 2\}$, the edge labelled a_i (b_i) is identified with the edge labelled a_i^{-1} (b_i^{-1}) in the direction shown.



The above construction can be generalized to every regular polygon whose number of sides is divisible by four. Anyway, some work is needed to make the above constructions precise. We begin by reviewing the quotient topology: Let X be a topological space and let $\sim$ be an equivalence relation on X. Further, let $X/\sim$ denote the set of equivalence classes, termed the *quotient* of X by $\sim$, and let $\pi : X \to X/\sim$ be the map which to every $x \in X$ associates its equivalence class. We turn $X/\sim$ into a topological space as follows: A set $U \subseteq X/\sim$ is open if $\pi^{-1}(U)$ is open. Using the fact that π respects all Boolean operations one verifies that this definition does indeed turn $X/\sim$ into a topological space. Furthermore, π is continuous.

Example 1.16. (Torus). Let $X := \mathbb{R}^2$. For all $x, y \in X$, set $x \sim y$ if and only if $x - y \in \mathbb{Z}^2$. Since $(\mathbb{Z}^2, +)$ is a group, this is indeed an equivalence relation. We are going to argue that $X/\sim$ is homeomorphic to the torus. First of all we determine a suitable piece of X which meets every equivalence class. The unit square $S := [0, 1]^2 \subset \mathbb{R}^2$ will serve. Indeed, let $\pi : X \to X/\sim$ denote the quotient map. Then $\pi([0, 1]^2) = X/\sim$. To see this, just floor the components of a vector $x \in X$ and subtract the resulting vector from x. However, it remains to be understood which points of S are identified under the equivalence relation. Clearly, if $x \in S$ is in the interior of S, no other point in S is equivalent to x. However, $(x_1, 0)^T \in S$ is identified with $(x_1, 1)^T \in S$ for all $x_1 \in [0, 1]$ and, similarly, $(0, x_2)^T \in S$ is identified with $(1, x_2)^T \in S$ for all $x_2 \in [0, 1]$. As observed above, these identifications yield a torus. The details of a homeomorphism are to be worked out.

The following example observes that not all equivalence relations yield reasonable quotient spaces.

Example 1.17. (A pathology). Let $X := \mathbb{R}$. For all $x, y \in X$, set $x \sim y$ if and only if $x - y \in \mathbb{Q}$. We show that $X/\sim$ is an uncountable space whose quotient topology has only the trivial open sets. Hence the topology is not suited at all to study the quotient in this case, it does not give any shape. First of all, $X/\sim$ is in fact uncountable since otherwise $\mathbb{R}$ would be a countable union of countable equivalence classes and thus countable. Now, let $U \subseteq X/\sim$ be open, i.e. $\pi^{-1}(U)$ is open. Hence $\pi^{-1}(U)$ contains a non-empty open interval (a, b). As a result, $(a, b) + \mathbb{Q} = \mathbb{R} \subseteq \pi^{-1}(U)$ since any real number can be approximated arbitrarily well by rationals. Hence the assertion.

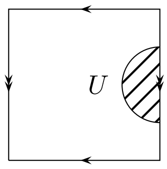
The last example shows that in order to obtain quotient topologies with good properties one has to impose some conditions on the equivalence relation.

Definition 1.18. Let X be a topological space. An equivalence relation $\sim$ on X is open if for every open set $U \subseteq X$, the image $\pi(U) \subseteq X/\sim$ is open.

Retain the notation of Definition 1.18. The statement is equivalent to demanding that the quotient map be open, or, by definition of the quotient topology, that $\pi^{-1}(\pi(U)) = \{y \in X \mid \exists x \in U : \ y \sim x\}$ be open in X for all open $U \subseteq X$.

Reviewing Example 1.16 in this regard we indeed see that the associated equivalence relation is open: Let $U \subseteq \mathbb{R}^2$ be open. Then $\pi^{-1}(\pi(U)) = \bigcup_{\gamma \in \mathbb{Z}^2}(\gamma + U)$ is open as a union of open sets.

However, note that non-open equivalence relations need not exclusively yield bad quotient spaces. In fact, consider $S = [0,1]^2 \subseteq \mathbb{R}^2$ with the induced topology and identify opposite sides via $\sim$ as in Example 1.16. Then $\sim$ is not open but the quotient space is the same as above: The image of the following open set $\mathcal{U} \subseteq S$ is not open. In a sense, one half of it is missing in the quotient.



In any case, open equivalence relations yield quotients which potentially satisfy the first two requirements on topological manifolds as follows. Recall that an equivalence relation $\sim$ on a set X is the subset $R := \{(x,y) \in X \times X \mid x \sim y\}$.

Proposition 1.19. Let X be topological space and let $\sim$ be an open equivalence relation on X.

(i) If X is second-countable then so is $X/\sim$.
(ii) The quotient $X/\sim$ is Hausdorff if and only if $R \subseteq X^2$ is closed.

Proof. As to (i), let $\mathcal{B}$ be a basis of open subsets of X and set $\mathcal{B}' := \{\pi(U) \mid U \in \mathcal{B}\}$. Since $\sim$ is open, $\mathcal{B}'$ consists of open sets. In addition, if $V \subseteq X/\sim$ is open then by definition so is $\pi^{-1}(V)$, hence $\pi^{-1}(V) = \bigcup_{U \in \mathcal{B},\ U \subseteq \pi^{-1}(V)} U$. Therefore

$$V = \pi(\pi^{-1}(V)) = \bigcup_{\substack{U \in \mathcal{B} \\ U \subseteq \pi^{-1}(V)}} \pi(U)$$

and hence $\mathcal{B}'$ is a basis as well.

Regarding (ii), first suppose that $X/\sim$ is Hausdorff. We show that $X \times X \backslash R$ is open. To this end, let $(x,y) \in X \times X \backslash R$, i.e. $\pi(x) \neq \pi(y)$. By assumption, there are open subsets U_x, U_y in $X/\sim$ containing $\pi(x)$ and $\pi(y)$ respectively such that $U_x \cap U_y = \emptyset$. Then $\pi^{-1}(U_x)$ and $\pi^{-1}(U_y)$ are open subsets of X containing x and y respectively. Hence $\pi^{-1}(U_x) \times \pi^{-1}(U_y)$ is an open subset of $X \times X$ containing (x,y). Finally, we observe that $\pi^{-1}(U_x) \times \pi^{-1}(U_y)$ does not intersect R. Indeed, suppose $(z_1, z_2) \in R \cap \pi^{-1}(U_x) \times \pi^{-1}(U_y)$. Then $\pi(z_1) = \pi(z_2)$ on the one hand and $\pi(z_1) \in U_x$ and $\pi(z_2) \in U_y$ on the other hand which contradicts $U_x \cap U_y = \emptyset$.

Conversely, assume that $R \subseteq X \times X$ is closed and let $x, y \in X$ such that $\pi(x) \neq \pi(y)$. Then $(x,y) \notin R$. Since R is closed there are open sets V_y and V_y containing x and y respectively such that $(V_x \times V_y) \cap R = \emptyset$. In other words, $\pi(V_x) \cap \pi(V_y) = \emptyset$. Conclude by recalling that $\sim$ is open and hence so are $\pi(V_x)$ and $\pi(V_y)$. $\square$

The second part of Proposition 1.19 can be applied to Example 1.16. Indeed, the equivalence relation

$$\begin{aligned} R &= \{(x,y) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid x - y \in \mathbb{Z}^2\} \\ &= \{(x,y) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid (e^{2\pi i(x_1-x_2)}, e^{2\pi i(y_1-y_2)}) = (1,1)\} \end{aligned}$$

is a level set of a continuous function and as such closed.

Example 1.20. (Real projective space). In linear algebra, $\mathbb{P}^n\,\mathbb{R}$ is defined as the quotient of $\mathbb{R}^{n+1}\setminus\{0\}$ by the equivalence relation $x \sim y$ if and only if $x = \lambda y$ for some $\lambda \in \mathbb{R}^*$. In fact, we could replace $\mathbb{R}$ with any other field. In this case, the equivalence relation is open and its graph R is closed. Hence $\mathbb{P}^n\,\mathbb{R} = \mathbb{R}^{n+1}\setminus\{0\}/\sim$ is second-countable and Hausdorff.

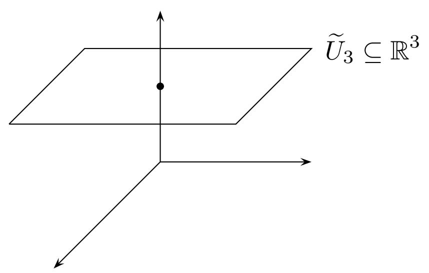
To see that $\sim$ is open, let $A \subseteq \mathbb{R}^{n+1}\setminus\{0\}$ be open, then $\pi^{-1}(\pi(A)) = \bigcup_{\lambda\in\mathbb{R}^*} \lambda A$ is open as well since multiplication by λ is a homeomorphism of $\mathbb{R}^{n+1}\setminus\{0\}$.

Now we show that R is closed. As before, we identify it as the level set of a continuous function. Define

$$f : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R},\ (x,y) \mapsto \sum_{i,j=1}^{n+1} (x_i y_j - x_j y_i)^2.$$

If $y = \lambda x$ then $y_i = \lambda x_i$ for all $i \in \{1,\dots,n+1\}$ and hence $x_i y_j - x_j y_i = x_i(\lambda x_j) - x_j(\lambda x_i) = 0$. Conversely, assume that $x, y \in \mathbb{R}^{n+1}\setminus\{0\}$ and $f(x,y) = 0$. Then $x_i y_j - x_j y_i = 0$ for all $i,j \in \{1,\dots,n+1\}$. Without loss of generality we may assume that $x_1 \neq 0$. Then $x_1 y_j - x_j y_1 = 0$ implies $y_j = (y_1/x_1)x_j$ for all $j \in \{1,\dots,n+1\}$, i.e. $y = \lambda x$ where $\lambda = y_1/x_1$. Since $y \neq 0$ we conclude that $\lambda \neq 0$.

We now equip $\mathbb{P}^n\,\mathbb{R}$ with charts: Set $\widetilde{U}_i := \{x = (x_1,\dots,x_{n+1}) \in \mathbb{R}^{n+1} \mid x_i = 1\}$.



Then $\pi^{-1}(\pi(\widetilde{U}_i)) = \{(x_1,\dots,x_{n+1}) \mid x_i \neq 0\}$ is open and hence by definition so is $U_i := \pi(\widetilde{U}_i) \subseteq \mathbb{P}^n\,\mathbb{R}$. Since $\pi|_{\widetilde{U}_i} : \widetilde{U}_i \to U_i$ is continuous by definition of the quotient topology, as well as injective and open, it is a homeomorphism. To build a chart, let $\mathrm{pr}_i : \mathbb{R}^{n+1} \to \mathbb{R}^n,\ x \mapsto (x_1,\dots,\hat{x}_i,\dots,x_{n+1})$ and observe that $\mathrm{pr}_i|_{\widetilde{U}_i} : \widetilde{U}_i \to \mathbb{R}^n$ is a homeomorphism. Hence $\varphi_i : U_i \to \mathbb{R}^n,\ y \mapsto \mathrm{pr}_i \circ (\pi|_{\widetilde{U}_i})^{-1}(y)$ is a homeomorphism as well. Combining this with the fact that $\bigcup_{i=1}^{n+1} U_i = \mathbb{P}^n\,\mathbb{R}$ shows that $(U_i,\varphi_i)_{i\in\{1,\dots,n+1\}}$ is an atlas on $\mathbb{P}^n\,\mathbb{R}$.

Next up, we compute the coordinate transformations. Let $1 \leq i < j \leq n+1$. Then $\varphi_i(U_i \cap U_j) = \{y = (y_1,\dots,y_n) \mid y_{j-1} \neq 0\}$ and $\varphi_j(U_i \cap U_j) = \{y = (y_1,\dots,y_n) \mid y_i \neq 0\}$. One then checks that $\theta_{ji} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ is given by

$$(y_1,\dots,y_n) \mapsto \frac{1}{y_{j-1}}(y_1,\dots,y_{i-1},1,y_i,\dots,\widehat{y}_{j-1},\dots,y_n)$$

and hence is smooth. Overall, we have shown that $\mathbb{P}^n\,\mathbb{R}$ admits the structure of a smooth n-manifold. Furthermore, it is compact: Simply note that $\pi|_{S_n} : S^n \to \mathbb{P}^n\,\mathbb{R}$

is surjective. (It is not injective, however; in fact every point in $\mathbb{P}^n\,\mathbb{R}$ has two preimages.

Example 1.21. (Grassmann manifolds). Grassmann manifolds generalize projective space and play a major role in the theory of vector bundles which we study later on. Let $1 \le k \le n$ and let $G(k,n)$ be the set of all subspaces of $\mathbb{R}^n$ of dimension k.

First of all, we describe a topology on $G(k,n)$ utilizing linear algebra: Let $F(k,n) := \{(v_1,\dots,v_k) \in (\mathbb{R}^n)^k \mid v_1,\dots,v_k \text{ linearly independent}\}$. Since every vector space admits a basis, we have a surjective map $\pi : F(k,n) \to G(k,n)$ given by $(v_1,\dots,v_k) \mapsto \mathbb{R}\,v_1 + \cdots + \mathbb{R}\,v_k$. This already puts us in a good position to define a topology: To k vectors $w_1,\dots,w_k$ of $\mathbb{R}^n$ we associate the $k \times n$-matrix

$$M(w_1,\dots,w_k) := \begin{pmatrix} w_1 \\ \vdots \\ w_k \end{pmatrix}.$$

Then $w_1,\dots,w_k$ are linearly independent if and only if $\operatorname{rank} M(w_1,\dots,w_k) = k$. In this way, $F(k,n)$ is identified with $\{A \in M_{k,n}(\mathbb{R}) \mid \operatorname{rank} A = k\}$, that is

$$\left\{ A \in M_{k,n}(\mathbb{R}) \;\middle|\; \exists 1 \le j_1 < \cdots < j_k \le n : \ \det \begin{pmatrix} A_{1j_1} & \cdots & A_{1j_k} \\ \vdots & & \vdots \\ A_{kj_1} & \cdots & A_{kj_k} \end{pmatrix} \neq 0 \right\}$$

which is union over multi-indices of open subsets of $M_{k,n}(\mathbb{R})$ and hence open. Furthermore, given $w_1,\dots,w_k$ and $w_1',\dots,w_k'$ one verifies that $\mathbb{R}\,w_1 + \cdots + \mathbb{R}\,w_k = \mathbb{R}\,w_1' + \cdots \mathbb{R}\,w_k'$ if and only if there is $B \in \mathrm{GL}(k,\mathbb{R})$ such that $M(w_1,\dots,w_k) = BM(w_1',\dots,w_k')$. Using this, one can show that $(w_1,\dots,w_k) \sim (w_1',\dots,w_k')$ if and only if $\mathbb{R}\,w_1 + \cdots \mathbb{R}\,w_k = \mathbb{R}\,w_1' + \cdots \mathbb{R}\,w_k'$ is an open equivalence relation: Indeed, given $B \in \mathrm{GL}(k,\mathbb{R})$, the map $L_B : M_{k,n}(\mathbb{R}) \to M_{k,n}(\mathbb{R}),\ A \mapsto BA$ is an invertible linear map with inverse $L_{B^{-1}}$. In particular, L_B is a homeomorphism. It preserves the open subset $F(k,n)$ and if $\Omega \subseteq F_{k,n}$ is open then so is $\pi^{-1}(\pi(\Omega)) = \bigcup_{B \in \mathrm{GL}(k,\mathbb{R})} L_B(\Omega)$. One may also show that the graph of $\sim$ is a closed subset of $F(k,n) \times F(k,n)$. We have thus equipped $G(k,n)$ with a second-countable Hausdorff topology.

We now define charts on $G(k,n)$ which turn it into a smooth manifold. Notice the analogy to the construction of Example 1.20. For any multi-index $J = (j_1,\dots,j_k)$ where $1 \le j_1 \le \cdots \le j_k \le n$, set

$$\widetilde{U}_J = \left\{ \begin{pmatrix} w_1 \\ \vdots \\ w_k \end{pmatrix} \in M_{k,n}(\mathbb{R}) \;\middle|\; \begin{pmatrix} w_{1j_1} & \cdots & w_{1j_k} \\ \vdots & & \vdots \\ w_{kj_1} & \cdots & w_{kj_k} \end{pmatrix} = \mathrm{Id} \in M_{k,k}(\mathbb{R}) \right\}.$$

Then $\pi|_{\widetilde{U}_J} : \widetilde{U}_J \to G(k,n)$ is injective and

$$\pi^{-1}(\pi(\widetilde{U}_J)) = \left\{ \begin{pmatrix} v_1 \\ \vdots \\ v_k \end{pmatrix} \;\middle|\; \det \begin{pmatrix} v_{1j_1} & \cdots & v_{1j_k} \\ \vdots & & \vdots \\ v_{kj_1} & \cdots & v_{kj_k} \end{pmatrix} \neq 0 \right\}.$$

Therefore, $\pi(\widetilde{U}_J) \subseteq G(k,n)$ is open and we have $\bigcup_J \pi(\widetilde{U}_J) = G(k,n)$. As a consequence $\pi|_{\widetilde{U}_J} : \widetilde{U}_J \to \pi(\widetilde{U}_J)$ is bijective, continuous and open, that is, a homeomorphism. In addition, we have the projection maps $p_J : \widetilde{U}_J \xrightarrow{\cong} M_{k,n-k}(\mathbb{R})$ and one verifies that $(\pi(\widetilde{U}_J), p_J \circ (\pi|_{\widetilde{U}_J})^{-1})_J$ is a smooth atlas on $G(k,n)$. We have $\dim G(k,n) = k(n-k)$.

1.3. **Differentiable Maps: Definition and Examples.** From the categorical viewpoint, we have so far introduced the objects. Morphisms, however, are still missing. In other words, without the means to compare smooth manifolds, nothing is going to happen.

Recall that in $\mathbb{R}^k$ we measure the length of vectors by $\|v\| = \sum_{i=1}^k v_i^2$. This is used in the definition of differentiability of a map defined on an open subset of $\mathbb{R}^n$.

Definition 1.22. Let $U \subseteq \mathbb{R}^n$ be open and let $p \in U$. A map $f : U \to \mathbb{R}^m$ is *differentiable at* $p \in U$ if there exists a linear map $L : \mathbb{R}^n \to \mathbb{R}^m$ such that $f(p+v) = f(p) + L(v) + R(p, v)$ for all $v \in \mathbb{R}^n$ such that $p + v \in U$ with

$$\lim_{v \to 0} \frac{\|R(p, v)\|}{\|v\|} = 0.$$

In the situation of Definition 1.22, we write $D_p f = L$. One of the most useful criteria for differentiability in $\mathbb{R}^n$ is the following.

Theorem 1.23. Let $U \subseteq \mathbb{R}^n$ be open. Further, let $f : U \to \mathbb{R}^m$, $x \mapsto (f_1(x), \ldots, f_m(x))$ be C^1 on U. Then f is differentiable at every point $p \in U$ and the matrix of $D_p f$ with respect to the standard bases on $\mathbb{R}^n$ and $\mathbb{R}^m$ is given by

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_n} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

Recall that in fact the existence of partial derivatives alone does not imply differentiability. For instance, the function

$$f : \mathbb{R}^2 \to \mathbb{R},\ (x, y) \mapsto \begin{cases} \frac{xy}{x^2+y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

has partial derivatives everywhere but is not differentiable at $(x, y) = (0, 0)$. In fact, it is not even continuous at this point.

We now define smooth functions on manifolds.

Definition 1.24. Let M be a smooth manifold and $p \in M$. A function $f : M \to \mathbb{R}$ is *differentiable at* p if for some chart (U, φ) at p, the function $\varphi(U) \to \mathbb{R},\ x \mapsto f \circ \varphi^{-1}(x)$ is differentiable at $\varphi(p)$.

Definition 1.24 is in fact independent of the chart chosen. The argument which shows this is an important and much used one: Let (U, φ) and (V, ψ) be charts including $p \in M$. For $x \in \psi(U \cap V)$ we compute $f \circ \psi^{-1}(x) = f \circ \varphi^{-1} \circ (\varphi \circ \psi^{-1})(x)$ which is a composition of the smooth map $\varphi \circ \psi^{-1}$ defined on $\psi(U \cap V)$ and $f \circ \varphi^{-1}$ which is differentiable at $\varphi(p)$. The same argument shows that the following definitions make sense.

Definition 1.25. Let M and N be smooth manifolds. Then

(i) $f \in C^k(M, \mathbb{R})$ if for every chart (U, φ) of M the function $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}$ is C^k,

(ii) $f : M \to N$ is differentiable at $p \in M$ if there are charts (U, φ) at p and (W, ψ) at $f(p)$ such that (1) $f(U) \subseteq W$ and (2) $\psi \circ f \circ \varphi^{-1} : \varphi(U) \to \psi(W)$ is differentiable at $\varphi(p)$,

(iii) $f \in C^k(M, N)$ if $f : M \to N$ is continuous and for every pair (U, φ), (W, ψ) of charts respectively on M and N such that $f(U) \subseteq W$ the map $\varphi(U) \to \psi(W),\ x \mapsto \psi \circ f \circ \varphi^{-1}(x)$ is C^k, and

(iv) $f : M \to N$ is a C^k-diffeomorphism if f is a homeomorphism and both f and f^{-1} are C^k-maps.

Retain Definition 1.25. Note that in (ii) we do not simply require (2) because without (1) it might be satisfied automatically for rather bad choices of f. Also, the reader is invited to check that Definition 1.25 is consistent, i.e. for instance that part (iii) for $N = \mathbb{R}$ yields the same notion as part (i). The following is an example of how results of calculus, namely Theorem 1.23 generalize to manifolds.

Theorem 1.26. Let M and N be manifolds and let $f : M \to N$ be a C^1-map. Then f is differentiable at every point $p \in M$.

Remark 1.27. We now collect several examples to illustrate Definition 1.25.

(i) Given a manifold M, the set $C^k(M, \mathbb{R})$ of C^k-functions on M is an $\mathbb{R}$-algebra for pointwise addition and multiplication of functions. This follows from the according statement for $M = \mathbb{R}$.
(ii) Let M, N and R be manifolds and let $f \in C^k(M, N)$ as well as $g \in C^k(N, R)$ be given. Then $g \circ f \in C^k(M, R)$ by the chain rule.
(iii) Let M and N be a manifolds and let $M = \bigcup_{\alpha \in A} U_\alpha$ be an open covering of M. Then $f \in C^k(M, N)$ if and only if $f|_{U_\alpha} \in C^k(U_\alpha, N)$ for all $\alpha \in A$, i.e. being C^k is a local condition.
(iv) Let $M = \mathbb{R}$ and $N = S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. Then $\exp : \mathbb{R} \to S^1$, $t \mapsto e^{2\pi i t}$ is smooth.
(v) Consider $\mathrm{GL}(n, \mathbb{R})$ with its smooth structure defined in Example 1.8. Then the map $\mathrm{GL}(n, \mathbb{R}) \times \mathrm{GL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})$, $(A, B) \mapsto AB$ is smooth. As a consequence, given $g \in \mathrm{GL}(n, \mathbb{R})$ the map $L_g : \mathrm{GL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})$, $a \mapsto ga$ is a diffeomorphism. In fact, a two-sided inverse of L_g is $L_{g^{-1}}$.

Remark 1.28. We remark further that any homeomorphism of a manifold can be turned into a smooth map: Let M be a smooth manifold with maximal atlas $\mathcal{A}$ and let $F : M \to M$ be a homeomorphism. Set $\mathcal{A}' = \{(F(U), \varphi \circ F^{-1}) \mid (U, \varphi) \in \mathcal{A}\}$. Then $\mathcal{A}'$ is a (maximal) smooth atlas on M and $F : (M, \mathcal{A}) \to (M, \mathcal{A}')$ is a smooth diffeomorphism.

We are now working towards the *rank theorem* for which we recall that the rank of a linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ is given by the dimension of its image which equals the row and column rank of any coordinate matrix of T.

Definition 1.29. Let M and N be manifolds, $p \in M$ and let $f : M \to N$ be differentiable at p. Further, let (U, φ) be a chart of M at p and (W, ψ) a chart of N at $f(p)$. The *rank of f at p* is the rank at $\varphi(p)$ of the linear map $D_{\varphi(p)}(\psi \circ f \circ \varphi^{-1}) : \mathbb{R}^n \to \mathbb{R}^m$.

The notion of rank of a map will be exploited a lot in the sequel. It leads to the concept of immersion, submersion, immersed manifolds etc. The key ingredient comes from the rank theorem of calculus to which we now turn.

1.4. **The Rank Theorem.**

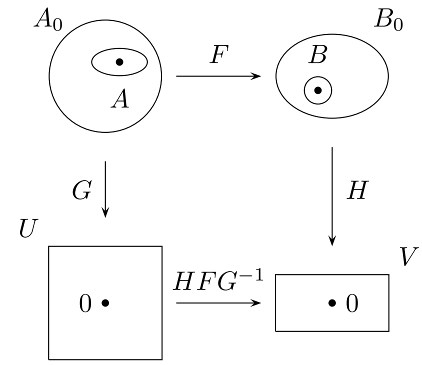
First of all, we recall the inverse function theorem.

Theorem 1.30. Let $W \subseteq \mathbb{R}^n$ be open and let $F : W \to \mathbb{R}^n$ be a C^r-map ($r \geq 1$) such that at $a \in W$, the derivative $D_a F : \mathbb{R}^n \to \mathbb{R}^n$ is invertible. Then there are open subsets U and V of $\mathbb{R}^n$ containing a and $F(a)$ respectively such that $F|_U : U \to V$ is a C^r-diffeomorphism.

This theorem is adaptable to the setting of smooth manifolds. However, for now we apply it to the following calculus result which says that up to smooth change of coordinates a map with constant rank k is the projection onto the first k components. To this end, recall the following fact from linear algebra: Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map of rank k. Note that necessarily $k \leq \min(m, n)$. Then there are invertible linear maps $G : \mathbb{R}^n \to \mathbb{R}^n$ and $H : \mathbb{R}^m \to \mathbb{R}^m$ such that $(H \circ L \circ G^{-1})(x_1, \ldots, x_n) = (x_1, \ldots, x_k, 0, \ldots, 0)$.

Theorem 1.31. Let $A_0 \subseteq \mathbb{R}^n$ and $B_0 \subseteq \mathbb{R}^m$ be open and let $F : A_0 \to B_0$ be a C^r-map ($r \geq 1$). Assume that F has constant rank k on A_0. Then given $a_0 \in A_0$ there are open subsets A and B of A_0 and B_0 respectively, containing a_0 and $F(a_0)$ respectively as well as C^r-diffeomorphisms $G : A \to U \subseteq \mathbb{R}^n$ and $H : B \to V \subseteq \mathbb{R}^m$ to open subsets U and V of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively such that $G(a_0) = 0$, $H(F(a_0)) = 0$ and $HFG^{-1}(x_1, \ldots, x_n) = (x_1, \ldots, x_k, 0, \ldots, 0)$.

The assertion of Theorem 1.31 may be visualized as follows.



We state that the change of variable maps G and H of Theorem 1.31 are not generally defined on the whole of A_0 and B_0.

Proof. Altering G and H with translations if necessary we may without loss of generality assume that $a_0 = 0$ as well as $F(a_0) = 0$. Now $D_0F = (\partial F_j / \partial x_i)_{i,j}$ where $1 \leq j \leq m$ and $1 \leq i \leq n$ has rank k. Modulo permuting coordinates in $\mathbb{R}^n$ and $\mathbb{R}^m$ we may assume that the first principal $k \times k$-minor of D_0F has non-zero determinant, that is

$$\det \left(\frac{\partial F_j}{\partial x_i} \right)_{\substack{1 \leq i \leq k \\ 1 \leq j \leq k}} \neq 0.$$

Define $G(x) := (F_1(x), \ldots, F_k(x), x_{k+1}, \ldots, x_n)$ where $F(x) = (F_1(x), \ldots, F_m(x))$. Observe that $G(0) = 0$ Furthermore, we have $x \in A_0$:

$$D_xG = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_k} & \frac{\partial F_1}{\partial x_{k+1}} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial F_k}{\partial x_1} & \cdots & \frac{\partial F_k}{\partial x_k} & \frac{\partial F_k}{\partial x_{k+1}} & \cdots & \frac{\partial F_k}{\partial x_n} \\ 0 & \cdots & 0 & 1 & & \\ \vdots & & \vdots & & \ddots & \\ 0 & \cdots & 0 & & & 1 \end{pmatrix} = \begin{pmatrix} \left(\frac{\partial F_i}{\partial x_j} \right) & * \\ 0 & \mathrm{Id}_{n-k} \end{pmatrix}.$$

That is, $\det D_xG = \det(\partial F_i / \partial x_j) \neq 0$. Hence we may apply the Inverse Function Theorem to G. There are open subsets $A_1 \subset A$ and $U_1 \subseteq \mathbb{R}^n$, containing 0 respectively such that $G : A_1 \to U_1$ is a C^r-diffeomorphism. Now, let us compute $F \circ G^{-1}$ on U_1: Let $y \in U_1$ and write $y = G(x)$ with $x \in A_1$. Then

$$\begin{aligned} F \circ G^{-1}(y) &= F \circ G^{-1}(F_1(x), \ldots, F_k(x), x_{k+1}, \ldots, x_n) = F(x_1, \ldots, x_n) \\ &= (F_1(x), \ldots, F_k(x), F_{k+1}(x), \ldots, F_m(x)) \\ &= (y_1, \ldots, y_k, f_{k+1}(y), \ldots, f_m(y)) \end{aligned}$$

for some $f_l(y) = F_l(x)$ with $l \geq k+1$. Now observe that $F \circ G^{-1}$ has constant rank k on U_1 and that for $y \in U_1$ we have $D_y(F \circ G^{-1}) = D_{G^{-1}(y)}F \circ D_yG$. Hence the lower right block in the matrix

$$D_y(F \circ G^{-1}) = \begin{pmatrix} \mathrm{Id}_k & 0 & \cdots & 0 \\ * & \frac{\partial f_{k+1}}{\partial y_{k+1}} & \cdots & \frac{\partial f_{k+1}}{\partial y_n} \\ \vdots & \vdots & & \vdots \\ * & \frac{\partial f_m}{\partial y_{k+1}} & \cdots & \frac{\partial f_m}{\partial y_n} \end{pmatrix}$$

vanishes. Modulo replacing U_1 with a smaller connected ε-ball we may thus assume that the functions $f_{k+1}, \dots, f_m$ only depend on $y_1, \dots, y_k$. Thus

$$F \circ G^{-1}(y_1, \dots, y_n) = (y_1, \dots, y_k, f_{k+1}(y_1, \dots, y_k), \dots, f_m(y_1, \dots, y_k))$$

Now define $T(z) := (z_1, \dots, z_k, z_{k+1} + f_{k+1}(z_1, \dots, z_k), \dots, z_m + f_m(z_1, \dots, z_k))$ for $z \in \mathbb{R}^m$ with $(z_1, \dots, z_k) \in \mathrm{pr}_1^k(U_1)$ where pr_1^k projects onto the first k coordinates. Then $T(0) = 0$ and

$$\begin{pmatrix} \mathrm{Id}_k & 0 \\ * & \mathrm{Id}_{n-k} \end{pmatrix}$$

is invertible. By the Inverse Function Theorem there is an open set $B_1 \subseteq \mathbb{R}^m$ containing 0 such that $T : B_1 \to T(B_1) \subseteq \mathbb{R}^m$ is a C^r-diffeomorphism onto its open image. Furthermore,

$$T(z_1, \dots, z_k, 0, \dots, 0) = (z_1, \dots, z_k, f_{k+1}(z_1, \dots, z_k), \dots, f_m(z_1, \dots, z_k))$$

which implies the assertion: $T^{-1}FG^{-1}(y_1, \dots, y_n) = (y_1, \dots, y_k, 0, \dots, 0)$. Now set $H = T^{-1}$. □

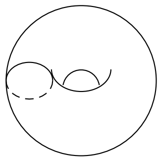
We remark that in $M_{m,n}(\mathbb{R})$ the condition of being of rank k is neither open nor closed and hence constitutes a strong assumption in the above theorem which utterly fails without it. As an immediate corollary of Theorem 1.31, we record that for a smooth map between manifolds there are always charts "adapted" to it. To this end we introduce the following notation: Given $a \in \mathbb{R}^n$ and $\varepsilon > 0$ set $C_\varepsilon^n(a) := \{x \in \mathbb{R}^n \mid \forall i \in \{1, \dots, n\} : |x_i - a_i| < \varepsilon\}$. The sets $C_\varepsilon^n(a)$ are hypercubes, for instance $C_\varepsilon^n(0) = (-\varepsilon, \varepsilon)^n$.

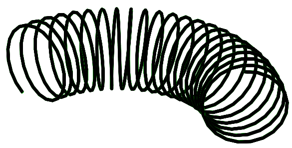
Corollary 1.32. Let N and M be manifolds of dimension n and m respectively and $p \in M$. Further, let $F : M \to N$ be a smooth map of constant rank k. Then there are charts (U, φ) at p and (W, ψ) at $F(p)$ such that $\varphi(p) = 0$, $\varphi(U) = C_\varepsilon^n(0) \subseteq \mathbb{R}^n$ as well as $\psi(F(p)) = 0$, $\psi(W) = C_\varepsilon^m(0)$ and

$$\psi F \varphi^{-1}(x_1, \dots, x_n) = (x_1, \dots, x_k, 0, \dots, 0)$$

for some $\varepsilon > 0$. □

1.5. **Submanifolds, Immersions, Embeddings etc.** We have defined the category of smooth manifolds and smooth maps. In some categories the image and kernel of a morphism are naturally identified as subobjects of the codomain and domain of the morphism. For instance, if $L : V \to W$ is a linear map between vector spaces then $L(V)$ is a subspace of W and $L^{-1}(0) \subseteq V$ is subspace of V. An analogous statement holds true in the category of groups. For smooth manifolds, however, images and level sets of smooth maps can be pretty bad. In the sequel we address both of these problems. First we establish conditions under which level sets of smooth maps are "nice". This leads us to introduce the notion of a submanifold. For instance, S^1 is a submanifold of the torus in e.g. the way depicted below. However, a real line wrapped around the torus at an irrational angle $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ as in $\iota : \mathbb{R} \to S^1 \times S^1$, $t \mapsto (e^{2\pi i t}, e^{2\pi i \alpha t})$ is not going to be a submanifold because a neighbourhood of a point in the image contains a dense set of image points. In fact, the image of ι is dense in $S^1 \times S^1$.





Definition 1.33. Let M be an m-manifold. A subset $N \subseteq M$ is an *n-submanifold* if for any $p \in N$ there is a chart (U, φ) at p such that $\varphi(p) = 0$, $\varphi(U) = C_\varepsilon^m(0) = (-\varepsilon, \varepsilon)^m$ and $\varphi(U \cap N) = \{x \in C_\varepsilon^m(0) \mid x_{n+1} = \cdots = x_m = 0\}$.

We remark that in the literature, Definition 1.33 often defines a *regular* submanifold. Also, one verifies easily that the restriction to N of all charts as in Definition 1.33 gives a smooth atlas on N and hence a structure of smooth n-manifold. Strictly speaking, we have the charts

$$\varphi|_{U\cap N} : U \cap N \to \{x \in C_\varepsilon^m(0) \mid x_{n+1} = \cdots = x_m = 0\} \xrightarrow{\mathrm{pr}_1^k} C_\varepsilon^n(0) \subseteq \mathbb{R}^n$$

Showing that the associated coordinate transformations are smooth amounts to saying that the restriction of a smooth map defined on $\mathbb{R}^n$ remains smooth when restricted to a coordinate plane.

Our results enable us to introduce the following mechanism to construct smooth manifolds.

Theorem 1.34. Let N and M be manifolds of dimension n and m respectively. Further, let $F : N \to M$ be a smooth map of constant rank k and let $q \in F(N)$. Then $F^{-1}(q) \subseteq N$ is a submanifold of N of dimension $n - k$.

Proof. Let $p \in F^{-1}(q) = \{x \in N \mid F(x) = q\}$. By Corollary 1.32 there are charts (U, φ) and (W, ψ) at p and $F(p)$ respectively such that $\varphi(p) = 0$, $\varphi(U) = C_\varepsilon^n(0)$, $\psi(W) = C_\varepsilon^m(0)$ and $\psi(q) = 0$ as well as $\psi F \varphi^{-1}(x_1, \ldots, x_n) = (x_1, \ldots, x_k, 0, \ldots, 0)$:

$$\begin{array}{ccc} U & \xrightarrow{F|_U} & W \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\psi} \\ C_\varepsilon^n(0) & \xrightarrow[\mathrm{pr}_1^k]{} & C_\varepsilon^m(0) \end{array}$$

In particular, $(F|_U)^{-1}(q) = U \cap F^{-1}(q)$ and $\varphi(U \cap F^{-1}(q)) = (\mathrm{pr}_1^k)^{-1}(0) = \{x \in C_\varepsilon^n(0) \mid x_1 = \cdots = x_k = 0\}$ which is a hypercube. Up to swapping coordinates this shows that $F^{-1}(q)$ is a submanifold in the sense of Definition 1.33. □

Using the following definitions we broaden the applicability of Theorem 1.34. In the sequel, N^n and M^m denote manifolds of dimension n and m respectively.

Definition 1.35. Let N^n and M^m be smooth manifolds and let $f : N \to M$ be a smooth map.

(i) A point $x \in N^n$ is a *critical point* if the rank of f at x is strictly less than m. The corresponding value $f(x)$ is *critical*.
(ii) A value $y \in M$ is *regular* if f has rank m at every point of $f^{-1}(y)$

Retain the notation of Definition 1.35. Notice that if $m \gneq n$ then every $x \in N$ is a critical point. We now have the following.

Theorem 1.36. Let N^n and M^m be smooth manifolds and let $f : N \to M$ be a smooth map. Further, let $y \in M$ be a regular value of f. Then $f^{-1}(y)$ is a regular $(n - m)$-dimensional submanifold of N.

Note that in Theorem 1.36, the set $f^{-1}(y)$ on which f has constant rank is a closed subset of N. Therefore, a priori, Theorem 1.34 is useless. However, we will exploit the fact that the rank at any point in $f^{-1}(y)$ is the maximal possible one using the following lemma.

Lemma 1.37. Let $n \geq m$ be integers. Then the set $R_m := \{A \in M_{m,n}(\mathbb{R}) \mid \operatorname{rank} A = m\}$ is open in $M_{m,n}(\mathbb{R})$.

Proof. For every $A \in R_m$ we construct an open neighbourhood of A contained in R_m. Let $1 \leq j_1 < \cdots < j_m \leq n$ be a multi-index such that the associated minor of A has non-zero determinant:

$$\det \begin{pmatrix} A_{1j_1} & \cdots & A_{1j_m} \\ \vdots & & \vdots \\ A_{mj_1} & \cdots & A_{mj_m} \end{pmatrix} \neq 0.$$

Then

$$V_A := \left\{ B \in M_{m,n}(\mathbb{R}) \,\middle|\, \det(B_{ij_l})_{\substack{1\leq l\leq m \\ 1\leq i\leq m}} \neq 0 \right\}$$

is an open subset of $M_{m,n}(\mathbb{R})$ containing A which is in fact contained in R_m by maximality of m since $\operatorname{rank} B \geq m$ for all $B \in V_A$. □

We now prove Theorem 1.36.

Proof. Consider the subset $R := \{x \in N \mid \operatorname{rank}_x f = m\}$ of N. By hypothesis, R contains $f^{-1}(y)$. Furthermore, R is open in N by Lemma 1.37: Let $x \in R$ and let (U, φ) and (W, ψ) be charts at x and $f(x)$ respectively such that $f(U) \subseteq W$. Consider $\varphi(U) \to \psi(W),\ z \mapsto \psi f \varphi^{-1}(z)$. We know that $\operatorname{rank} D_{\varphi(x)}(\psi f \varphi^{-1}) = m$. In addition, the map $\varphi(U) \to M_{n,n}(\mathbb{R})$ which to $z \in \varphi(U)$ associates $D_z(\psi f \varphi^{-1})$ is continuous. Hence, by Lemma 1.37, the set $\{z \in \varphi(U) \mid \operatorname{rank} D_z(\psi f \varphi^{-1}) = m\}$ is open and contains $\varphi(x)$. Therefore, there is an open subset of U containing x on which f has constant rank m which implies that R is open. Thus R is a smooth n-manifold on which f has constant rank. Applying Theorem 1.34 yields the assertion. □

Given that individual level sets of a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ may be arbitrary closed subsets of $\mathbb{R}^n$, one would not expect regular values to be abundant. However, we shall see in Theorem 1.43 that in fact they are in a very precise sense. First of all, though, we give a sample application of Theorem 1.36.

Theorem 1.38. The orthogonal group $\mathrm{O}(n) := \{X \in \mathrm{GL}(n, \mathbb{R}) \mid X^T X = \mathrm{Id}\}$ is a regular submanifold of $\mathrm{GL}(n, \mathbb{R})$ of dimension $n(n-1)/2$: To see this, consider the map $F : M_{n,n}(\mathbb{R}) \to M_{n,n}(\mathbb{R})$ given by $X \mapsto X^T X$. We show that F has constant rank $n(n-1)/2$ on $\mathrm{GL}(n, \mathbb{R})$. We compute

$$D_X F(Y) = \lim_{h\to 0} \frac{F(X + hY) - F(X)}{h} = Y^T X + X^T Y$$

and note that the map

$$D_X F : M_{n,n}(\mathbb{R}) \to M_{n,n}(\mathbb{R}),\ Y \mapsto Y^T X + X^T Y$$

ranges in symmetric matrices. We therfore reconsider

$$D_X F : M_{n,n}(\mathbb{R}) \to \mathrm{Sym}_n(\mathbb{R}),\ Y \mapsto Y^T X + X^T Y$$

and note that if X is invertible then the image of $D_X F$ is $\mathrm{Sym}_n(\mathbb{R})$. Hence F has constant rank $n(n+1)/2$ on $\mathrm{GL}(n, \mathbb{R})$ which implies the assertion.

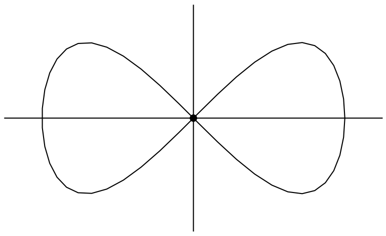
Having discussed level sets of smooth functions we now turn to the question under which circumstances the image of a smooth map between manifolds is a manifold.

Definition 1.39. Let N^n and M^m be smooth manifolds and let $f : N \to M$ be a smooth map. Then f is an *immersion* if the rank of f at every point is n.

Observe that an immersion $f : N^n \to M^m$ is only possible for $n \leq m$. In local coordinates, an immersion looks nice by Theorem 1.34. Also, the inverse image of a point under f is a 0-dimensional submanifold of N and hence discrete.

Example 1.40. We give two examples of an immersion.

(i) Let $f : \mathbb{R} \to \mathbb{R}^2,\ t \mapsto (2\cos t, \sin 2t)$. Since $\dot{f}(t) = (-2\sin t, 2\cos 2t) \neq 0$ for all $t \in \mathbb{R}$ this map is in fact an immersion, of constant rank one. Its graph looks as follows:



Notice that $f(\pi/2) = (0,0)^T = f(3\pi/2)$. In particular, immersions need not be injective. However, the tangent vectors of f at these two points differ.

(ii) As before, consider the irrationally imbedded real line into the torus given by $f : \mathbb{R} \to S^1 \times S^1,\ t \mapsto (e^{2\pi i t}, e^{2\pi i \alpha t})$ where $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Whereas f is injective, its image is dense which is considered bad for reasons that will become clear later.

The following is a stronger notion than immersion.

Definition 1.41. Let N^n and M^m be smooth manifolds and let $f : N \to M$ be a smooth map. Then f is an *embedding* if $f(N) \subseteq M$ is a regular submanifold and if $f : N \to f(N)$ is a diffeomorphism.

Proposition 1.42. Let N^n and M^m be smooth manifolds and let $f : N \to M$ be a smooth map. If f is an immersion and a homeomorphism onto its image then f is an embedding.

This follows essentially from Theorem 1.34.

Proof. Since $f : N \to M$ is a homeomorphism onto its image we know that for every open subset $V \subseteq N$ there is an open subset $W \subseteq M$ with $f(V) \subseteq W \cap f(N)$. Now choose charts (V, φ) and (W, ψ) at p and $f(p)$ respectively such that $f(V) \subseteq W \cap f(N)$ and $\varphi(V) = C^n_\varepsilon(0)$, $\varphi(p) = 0$ as well as $\psi(W) = C^m_\varepsilon(0)$, $\psi(f(p)) = 0$ and

$$\psi f \varphi^{-1} : C^n_\varepsilon(0) \to C^m_\varepsilon(0),\ x \mapsto (x, 0).$$

$$\psi(W \cap f(N)) = \{y \in C^m_\varepsilon(0) \mid y_{n+1} = \cdots = y_m = 0\}$$

which shows that $f(N)$ is a regular submanifold of M. ∎

1.6. **Sard's Theorem.** We now turn to the arguably most important theorem in Differential Topology. While individual level sets of smooth maps can be very bad, there are quite general situations in which a smooth map has lots of regular values; this follows from Sard's Theorem which says that in any case the image of the set of critical points is always small.

Theorem 1.43 (Sard, 1942). Let $U \subseteq \mathbb{R}^n$ be open and let $f : U \to \mathbb{R}^m$ be smooth. Let $C := \{x \in U \mid \operatorname{rank} D_x f < m\}$ be the set of critical points of f. Then $f(C) \subseteq \mathbb{R}^m$ has Lebesgue measure zero.

A few remarks about this theorem are in order. First of all, recall that a subset $E \subseteq \mathbb{R}^m$ has measure zero if for every $\varepsilon > 0$ there is a covering $E \subseteq \bigcup_{i=1}^{\infty} C_i$ by countably many hypercubes such that $\sum_{i=1}^{\infty} \operatorname{vol}(C_i) < \varepsilon$. We will use several times the fact that a countable union of sets of measure zero has measure zero. This follows from the above definition by choosing geometrically decaying values for the respective ε.

For instance, this implies the disturbing fact that $\mathbb{Q} \subseteq \mathbb{R}$ has measure zero: Indeed, let $f : \mathbb{N} \to \mathbb{Q}$ be a bijection. Then $f(n) \in (f(n) - \varepsilon/2^n, f(n) + \varepsilon/2^n)$ and the sum of the volumes of these intervals is 2ε.

Also, note the following: Let $f : [0,1] \to \mathbb{R}$ be monotonically increasing, i.e. $f(t_1) \leq f(t_2)$ whenever $t_1 \leq t_2$. Then the set of points at which f is not differentiable is of measure zero. It therefore seems like one could deduce a lot of information about f using its derivative; however, there is a monotonically increasing f whose derivative is zero whenever it exists. See [RSN72] for a nice introduction to examples of this sort.

Remark 1.44. We now discuss Theorem 1.43 in the two cases $n < m$ and $n \geq m$ because the flavour is somewhat different.

(i) ($n < m$). In this case, $C = U$ and hence the statement is that $f(U)$ is of measure zero in $\mathbb{R}^m$. As a warm-up to this, we prove that if $U \subseteq \mathbb{R}$ is open and $f : U \to \mathbb{R}^m$ for $m \geq 2$ is C^1 then $f(U)$ has measure zero. First of all, note that the C^1 assumption is indeed necessary as there is a continuous surjective map $f : [0,1] \to [0,1]^2$. By reparameterization, assume that $[0,1] \subseteq U$. Then there is a constant $C > 0$ such that $\|f(x) - f(y)\| \leq C|x - y|$ for all $x, y \in [0,1]$. Given $k \in \mathbb{N}$, subdivide the interval $[0,1]$ into the subintervals $[j/k, (j+1)/k]$. Then $f([j/k, (j+1)/k])$ is contained in a hypercube of side length at most C/k and $f(C)$ can be covered by k hypercubes of this side length. The volume of each such hypercube is $(C/k)^m$ and hence the sum of their volumes is at most $k(C/k)^m = C^m k^{1-m}$. Since $m \geq 2$, this tends to zero as k tends to infinity.

(ii) ($n \geq m$). Assume that $f : U \to \mathbb{R}^m$ is defined on an open subset $U \subseteq \mathbb{R}^n$. Also, let us assume that $C \neq U$, that is, there is a regular point $x \in U$. Then there is an open subset V in U containing x on which f has maximal rank. Hence $f(V) \subseteq \mathbb{R}^m$ is open and as a consequence cannot have measure zero. Hence $f(U) \backslash f(C)$ is non-empty. In particular, there are tons of regular values! In fact, Sard's Theorem 1.43 is often applied by stating the existence of a regular value.

(iii) We remark that Sard's Theorem remains true under the weaker assumption that f be C^k for some $k \geq \max\{1, n - m + 1\}$. Thus for $n = 1$ it suffices in fact to ask for f to be C^1 whereas when $m = 1$ the function f needs to be C^n. For instance, there is an example of a C^1-function $f : \mathbb{R}^2 \to \mathbb{R}$ whose image contains an interval of singular values.

We now turn to the proof of Theorem 1.43. It contains two major ideas.

Proof. The proof proceeds by induction on n. Note that the statement makes sense for $n \geq 0$ and $m \geq 1$. For $n = 0$ and $m \geq 1$ the theorem is true since the image of the one-point space $\mathbb{R}^0$ has measure zero in $\mathbb{R}^m$ for $m \geq 1$. Now, let $n \geq 1$ and write $f = (f_1, \ldots, f_m)$. Recall that $C = \{x \in U \mid \operatorname{rank} D_x f < n\}$. Set

$$C_1 := \{x \in U \mid D_x f \equiv 0\} = \left\{ x \in U \,\middle|\, \frac{\partial f_r}{\partial x_i}(x) = 0 \;\forall r \in \{1, \ldots, m\} \;\forall i \in \{1, \ldots, n\} \right\}.$$

More generally, for $i \geq 1$, let C_i denote the set of all $x \in U$ for which all partial derivatives of f up to order i vanish at x. Then

$$\cdots \subseteq C_{i+1} \subseteq C_i \subseteq \cdots \subseteq C_1 \subseteq C.$$

The proof is now divided into the follwing three steps.

(i) Show that $f(C \backslash C_1)$ has measure zero.

(ii) Show that $f(C_i \backslash C_{i+1})$ has measure zero for all $i \geq 1$.

These two steps are not yet enough to conclude the assertion as we do not know about the measure of $\bigcap_{i \geq 1} C_i$. We therefore end with the following step.

(iii) Show that $f(C_k)$ has measure zero for large enough k. In fact, $k > n/m - 1$.

The first two steps are based on the same idea whereas the third one introduces a new one.

Step (i). We consider $C \backslash C_1$. Observe that for $m = 1$ we have $C = C_1$. Indeed, for a function $f : U \to \mathbb{R}$ the condition that $\operatorname{rank} D_x f < 1$ is equivalent to the vanishing of all first partial derivative at x. Hence $f(C \backslash C_1)$ is empty. We may thus assume $m \geq 2$. In this situation, let $\overline{x} \in C \backslash C_1$. By definition and without loss of generality we may thus assume $(\partial f_1 / \partial x_1)(\overline{x}) \neq 0$. Define a change of variables $h : U \to \mathbb{R}^n$ by $x \mapsto (f_1(x), x_2, \ldots, x_n)$ and observe that

$$D_{\overline{x}} h = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \cdots & \frac{\partial f_1}{\partial x_m} \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

which has non-zero determinant. Hence, by the Inverse Function Theorem, there is an open subset V of U containing $\overline{x}$ and an open subset $V' \subseteq \mathbb{R}^n$ containing $h(\overline{x})$ such that $h|_V : V \to V'$ is a diffeomorphism. Now define $g := f \circ (h|_V)^{-1} : V' \to \mathbb{R}^m$. Let $C' \subseteq V'$ be the set of critical points of g. Then the chain rule implies that $h(V \cap C) = C'$. Hence $f(V \cap C) = g(C')$. It now suffices to show that $g(C')$ has measure zero since we can cover $U \backslash C_1$ with countably many such V's and deduce that $f(C \backslash C_1)$ has measure zero.

As in Theorem 1.34, we show that g has a special form. Let $(t, x_2, \ldots, x_n) = (f_1(x), x_2, \ldots, x_n) = h(x) \in V'$. Then

$$\begin{aligned} g(t, x_2, \ldots, x_n) = gh(x) &= f(x) \\ &= (f_1(x), \ldots, f_m(x)) \\ &= (t, f_2 h^{-1}(t, x_2, \ldots, x_n), \ldots, f_m h^{-1}(t, x_2, \ldots, x_n)) \\ &= (t, g^t(x_2, \ldots, x_n) \end{aligned}$$

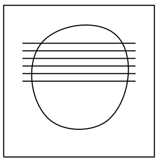
where $g^t : \mathbb{R}^{n-1} \to \mathbb{R}^{m-1}$ is given by

$$g^t(x_2, \ldots, x_n) := (f_2 h^{-1}(t, x_2, \ldots, x_n), \ldots, f_m h^{-1}(t, x_2, \ldots, x_n).$$

Observe that $n - 1 \geq 0$ and $m - 1 \geq 1$ which is covered by the induction hypothesis. Compute

$$D_{(t, x_2, \ldots, x_n)} g = \begin{pmatrix} 1 & 0 \\ * & D_{(x_2, \ldots, x_n)} g^t \end{pmatrix}.$$

Therefore, $(t, x_2, \ldots, x_n)$ is critical for g if and only if $(x_2, \ldots, x_n)$ is critical for g^t. Let C^t denote the critical points of g^t. Then $g(C') = \{(t, z) \mid z \in g^t(C_t\}$ and hence $g(C') \cap (\{t\} \times \mathbb{R}^{m-1}) = \{t\} \times g^t(C_t)$. By the induction hypothesis, the Lebesgue measure of $g^t(C_t)$ is zero for all t and hence by Fubini's Theorem $g(C')$ has measure zero as well. For an efficient introduction to measure theory, see [Rud87].



This completes step one.

Step (ii). Let $i \geq 1$ and $\overline{x} \in C_i \backslash C_{i+1}$. Then for some $1 \leq r \leq m$ and some multi-index $1 \leq j_1 \leq \cdots \leq j_{i+1} \leq n$ we have

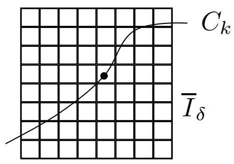
$$w(\overline{x}) := \frac{\partial^i f_r(\overline{x})}{\partial x_{j_2} \cdots \partial x_{j_{i+1}}} = 0$$

but $\frac{\partial w}{\partial x_{j_1}} \neq 0$. Without loss of generality, assume $j_1 = 1$, that is $\frac{\partial w}{\partial x_1} \neq 0$. Again, consider the map $h : U \to \mathbb{R}^n$ given by $x \mapsto (w(x), x_2, \ldots, x_n)$ which is a diffeomorphism from some open set $V \subseteq U$ containing $\overline{x}$ onto some open set $V' \subseteq \mathbb{R}^n$. By definition, $h(C_i \cap V) \subseteq (\{0\} \times \mathbb{R}^{n-1})$. Now, consider again the map $g := f \circ h^{-1} : V' \to \mathbb{R}^m$. Then $gh(C_i \cap V) = f(C_i \cap V)$ but since $h(C_i \cap V) \subseteq \{0\} \times \mathbb{R}^{n-1}$ we may consider $\overline{g} := g|_{\{0\} \times \mathbb{R}^{n-1}}$. Observe that any point in $h(C_i \cap V)$ is certainly a critical point of $\overline{g}$. Hence we are done by recurrence.

Step (iii). We now show for large enough k, the set $f(C_k)$ has measure zero. To this end, let $\overline{I}_\delta \subseteq U$ be a closed hypercube whose edges have length δ. We show that $f(C_k \cap \overline{I}_\delta)$ has measure zero for $k > n/m - 1$. By Taylor, we have for all $x \in C_k \cap \overline{I}_\delta$ and h with $x + h \in \overline{I}_\delta$ that

$$f(x + h) = f(x) + R(x, h)$$

where $\|R(x, h)\| \leq C\|h\|^{k+1}$ and C only depends on f and $\overline{I}_\delta$. More precisely, this is a consequence of the integral form of the remainder. Now, pick N to subdivide $\overline{I}_\delta$ into cubes of side length δ/N. Let I be one of these small cubes containing a point $x \in C_k \cap I$.



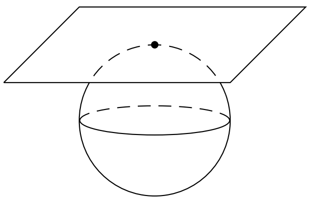
Then for all h such that $x + h \in I$ we have

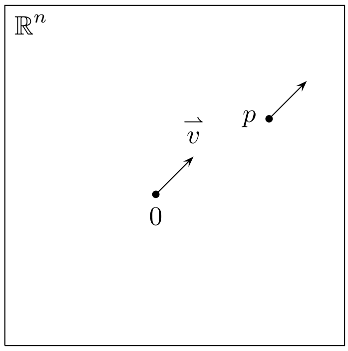
$$\|f(x + h) - f(x)\| \leq C\|h\|^{k+1} \leq C \left(\sqrt{n} \frac{\delta}{N} \right)^{k+1}.$$

Hence $f(I)$ is contained in a hypercube with edges of the above length. There are at most N^n such little cubes and hence $f(C_k \cap \overline{I}_\delta)$ is contained in a union of hypercubes whose sum of volumes is less than $N^n(C\sqrt{n}\delta/N)^{(k+1)m} = (C\sqrt{n}\delta)^{(k+1)m} \cdot N^{n-(k+1)m}$. For $k > n/m - 1$ we can make this sum arbitrarily small by choosing N large enough. Hence the assertion. □

2. Tangent spaces, Differential and Whitney's Embedding Theorem

2.1. **Tangent Spaces.** The naive idea of a tangent "plane" approximating a surface up to order one cannot be implemented in general for lack of an ambient space. However, there are at least two ways to define a tangent space and recover the above intuition. We will start from the following viewpoint: Let M and N be manifolds and $p \in M$. The tangent space T_pM should be equal to $\mathbb{R}^m$ if $M = \mathbb{R}^m$, and such that the construction is functorial, meaning that the derivative D_pf is a linear map from $\mathrm{T}_pN \to \mathrm{T}_{f(p)}M$.

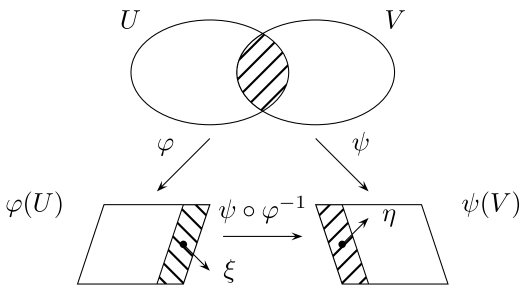




The mental picture of course is that a tangent vector $\vec{v}$ at a point p is attached to p as it represents a small increment of the position vector of p. Now, let $\mathcal{A}$ be an atlas on M and define

$$\mathcal{A}_p := \{(U, \varphi, \xi) \mid (U, \varphi) \text{ is a chart at } p,\ \xi \in \mathbb{R}^m\}$$

On $\mathcal{A}_p$ we define a relation $\sim_p$ as follows. Set $(U, \varphi, \xi) \sim_p (V, \psi, \eta)$ if and only if $D_{\varphi(p)}(\psi \circ \varphi^{-1})\xi = \eta$.



Lemma 2.1. Retain the above notation. The relation $\sim_p$ is an equivalence relation.

Proof. Clearly, $\sim_p$ is reflexive. As to symmetry, suppose $(U, \varphi, \xi) \cong (V, \psi, \eta)$, that is $D_{\varphi(p)}(\psi\varphi^{-1})\xi = \eta$. Hence $\xi = (D_{\varphi(p)}(\psi\varphi^{-1}))^{-1}(\eta)$. However, by the chain rule, $(D_{\varphi(p)}(\psi\varphi^{-1}))^{-1} = D_{\psi(p)}(\varphi\psi^{-1})$ and hence $\xi = D_{\psi(p)}(\varphi\psi^{-1})\eta$ which is the assertion. For transitivity, suppose $(U_1, \varphi_1, \xi_1) \cong (U_2, \varphi_2, \xi_2)$ and $(U_2, \varphi_2, \xi_2) \cong (U_3, \varphi_3, \xi_3)$. Then $D_{\varphi_1(p)}(\varphi_2\varphi_1^{-1})\xi_1 = \xi_2$ and $D_{\varphi_2(p)}(\varphi_3\varphi_2^{-1})\xi_2 = \xi_3$. Now observe that on $\varphi_1(U_1 \cap U_2 \cap U_3)$, which contains $\varphi_1(p)$ we have $(\varphi_3\varphi_2^{-1})(\varphi_2\varphi_1^{-1}) = \varphi_3\varphi_1^{-1}$. Hence the chain rule implies that

$$D_{\varphi_1(p)}(\varphi_3\varphi_1^{-1}) = D_{\varphi_2(p)}(\varphi_3\varphi_2^{-1}) \circ D_{\varphi_1(p)}(\varphi_3\varphi_1^{-1})$$

maps ξ_1 to ξ_3 which is the assertion. $\square$

Given the above lemma, we now define $\mathrm{T}_pM := \mathcal{A}_p \,/ \sim_p$. Hence a tangent vector at p is a consistent choice of tangent vectors in the charts. Next, we record that T_pM does indeed admit a natural vector space structure.

Lemma 2.2. Let M be a manifold and let (U, φ) be a chart at $p \in M$. Then the map $\mathbb{R}^m \to \mathrm{T}_pM,\ \xi \mapsto [(U, \varphi, \xi)]$ is a bijection. The resulting vector space structure on T_pM is independent of the choice of the chart (U, φ).

Proof. First, we show injectivity: Let $\xi_1, \xi_2 \in \mathbb{R}^m$ and suppose that $(U, \varphi, \xi_1) \sim_p (U, \varphi, \xi_2)$, i.e. $D_{\varphi(p)}(\varphi \circ \varphi^{-1})\xi_1 = \xi_2$ which implies $\xi_1 = \xi_2$. As to surjectivity: Let $v = [(V, \psi, \eta)] \in \mathrm{T}_pM$. Then $(U, \varphi, D_{\psi(p)}(\varphi \circ \psi^{-1})\eta)$ is equivalent to v.

Regarding the vector space structure, we need to show that addition and scalar multiplication defined with respect to different choices of (U, φ) coincide. To this end, let $v_1, v_2 \in \mathrm{T}_pM$. Assume that v_i ($i \in \{1, 2\}$) is represented by both (U, φ, ξ_i) and (V, ψ, η_i). We need to show that $(U, \varphi, \xi_1 + \xi_2) \sim_p (V, \psi, \eta_1 + \eta_2)$. Indeed, we have $D_{\varphi(p)}(\psi\varphi^{-1})\xi_1 = \eta_1$ and $D_{\varphi(p)}(\psi\varphi^{-1})\xi_2 = \eta_2$ and therefore $D_{\varphi(p)}(\psi\varphi^{-1})(\xi_1 + \xi_2) = \eta_1 + \eta_2$ which amounts to the assertion. A similar argument works for scalar multiplication. □

Remark 2.3. With this definition of tangent space we indeed recover the following: Let $U \subseteq \mathbb{R}^m$ be an open subset, considered as a smooth m-manifold. Then using the chart (U, Id) at any point $p \in U$ leads to the identification $\mathrm{T}_pU = \mathbb{R}^m$.

Now, let $f : N \to M$ be a map which is differentiable at p. We define the *differential* $D_pf : \mathrm{T}_pN \to T_{f(p)}M$ *of* f *at* p in the following way: Pick local charts (U, φ) at p and (V, ψ) at $f(p)$ such that $f(U) \subseteq V$. Given $v := [(U, \varphi, \xi)] \in \mathrm{T}_pN$ we define $D_pf(v) := [(V, \psi, D_{\varphi(p)}(\psi f \varphi^{-1})\xi)]$. Recall that by definition of smoothness of f at p the map $\psi f \varphi^{-1}$ is indeed smooth at $\varphi(p)$. The proof of the following lemma is now left as an exercise.

Lemma 2.4. Retain the above notation. The map $D_pf : \mathrm{T}_pN \to \mathrm{T}_{f(p)}M$ is a well-defined linear map. Further, the rank of f at p equals $\dim D_pf(\mathrm{T}_pN)$.

Also, the chain rule is generalized to the setting of manifolds.

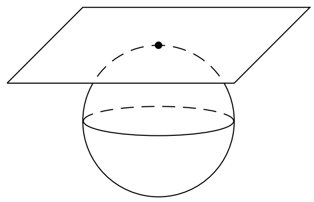
Lemma 2.5. Let N, M and R be manifolds and $f : N \to M$ as well as $g : M \to R$ be smooth maps. If f is differentiable at $p \in N$ and g is differentiable at $f(p) \in M$ then $gf : M \to R$ is differentiable at $p \in N$ and $D_pgf = D_{f(p)}g \circ D_pf$.

The following proposition underlines once more that our definition captures the intuition: Let N, M be manifolds and $f : N \to M$ a smooth map of constant rank k. Then for $y \in f(N)$, the set $f^{-1}(y) \subseteq N$ is a regular $(n-k)$-submanifold. Let us consider $f^{-1}(y)$ as an $(n-k)$-manifold and the injection $i : f^{-1}(y) \hookrightarrow N$ which is of rank $n - k$. Then for all $x \in f^{-1}(y)$ we have the map $D_xi : \mathrm{T}_x(f^{-1}(y)) \to \mathrm{T}_{i(x)}N$.

Proposition 2.6. Retain the above notation. For every $x \in f^{-1}(y)$ we have

$$D_xi(T_xf^{-1}(y)) = \ker D_xf.$$

The reader is encouraged to verify that this captures the intuition from above in the case $N = \mathbb{R}^3 \setminus \{0\}$, $M = (0, \infty)$, $f(x) = \|x\|^2$, $y = 1$ and $f^{-1}(y) = S^2$:



Proof. This can be proven directly by unravelling the definitions but there is also a way to avoid that. Consider the constant map $f \circ i : f^{-1}(y) \to M,\ x \mapsto y$. Then $D_x(f \circ i) = 0$. Using the chain rule we deduce $D_{i(x)} f \circ D_x i = 0$. In particular, $D_x i(\mathrm{T}_x f^{-1}(y)) \subseteq \ker(D_{i(x)} f)$. For the other inclusion, observe that rank of i at any point $x \in f^{-1}(y)$ is $n - k$. Hence $\dim D_x i(\mathrm{T}_x f^{-1}(y)) = n - k$. Since the rank of f at any point is k we conclude that $\dim \ker D_{i(x)} f = n - k$ because $\dim \mathrm{T}_{i(x)} N - \dim \ker D_{i(x)} f = \dim D_{i(x)} f = k$ and $\dim T_{i(x)} N = n$. This implies the converse inclusion. □

2.2. **Tangent Vectors and Derivations.** In this section, we discuss an equivalent, more algebraic definition of the tangent space which is particularly useful e.g. in the setting of Lie groups. Let N be an n-manifold and let $p \in N$. Recall that $C^\infty(N) := C^\infty(N, \mathbb{R})$ is an $\mathbb{R}$-algebra.

Definition 2.7. Let N be a manifold and $p \in N$. A *derivation of* $C^\infty(N)$ *at* p is a map $\delta : C^\infty(N) \to \mathbb{R}$ such that

(i) δ is an $\mathbb{R}$-linear map, and
(ii) (Leibniz rule) for all $f, g \in C^\infty(N)$ we have $\delta(fg) = \delta(f)g(p) + f(p)\delta(g)$.

Let $\mathrm{Der}_p\, C^\infty(N)$ be the vector space of derivations at p

Theorem 2.8. Let N be a manifold and $p \in N$. The map

$$\mathrm{T}_p N \to \mathrm{Der}_p\, C^\infty(N),\ v \mapsto (\delta_v : f \mapsto D_p f(v))$$

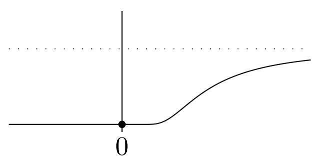
is an isomorphism of vector spaces.

Note that Definition 2.7 is much easier to write down as a definition of the tangent space but is also a lot less transparent. Nevertheless, as noted above, it is an important description of the tangent space. We remark that the C^∞-assumption in Definition 2.7 is crucial. There are much more derivations of $C^k(N)$ than those which come from tangent vectors.

The strategy of proof of Theorem 2.8 is to translate the problem to $\mathbb{R}^n$ using charts, solve it there and translate back the solution. The following lemmas are concerned with the Euclidean setting.

In particular, the following function is needed in the proof: Let $\sigma : \mathbb{R} \to \mathbb{R}$ be defined by

$$\sigma(x) := \begin{cases} e^{-\frac{1}{x^2}} & x > 0 \\ 0 & x \leq 0 \end{cases}.$$

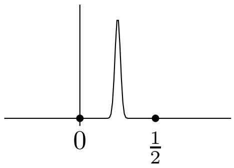


Then $\sigma \in C^\infty(\mathbb{R})$.

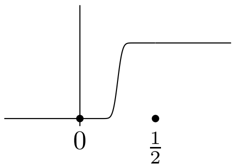
Lemma 2.9. Let $K \subseteq \mathbb{R}^n$ be compact and let $F \subseteq \mathbb{R}^n$ be closed with $K \cap F = \emptyset$. Then there is $f \in C^\infty(\mathbb{R}^n)$ such that $0 \leq f(x) \leq 1$ for all $x \in \mathbb{R}^n$, $f|_K \equiv 1$ and $f|_F \equiv 0$.

Proof. We utilize the following more precise statement which will also be used in the proof of Whitney's embedding theorem: Let $a \in \mathbb{R}^n$ and $\varepsilon > 0$. Then there is $f \in C^\infty(\mathbb{R}^n)$ such that $0 \leq f(x) \leq 1$ for all $x \in \mathbb{R}^n$ as well as $f^{-1}(1) = \overline{C^n_{\varepsilon/2}(a)}$ and $f^{-1}(0) = \mathbb{R}^n \setminus C^n_\varepsilon(a)$.

First, recall the function σ from above and consider the map $\mathbb{R} \to \mathbb{R}$ which sends t to $\sigma(t)\sigma(1/2 - t)$ and whose graph looks as follows:



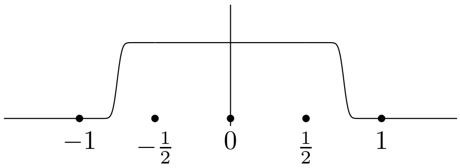
Next, consider the map $\mathbb{R} \to \mathbb{R}$, $x \mapsto \int_{-\infty}^{x} \sigma(t)\sigma(1/2 - t)\ dt$. One easily determines that its graph looks as follows:



Finally, normalize the above function so that the constant value which it assumes on $[1/2, \infty)$ is one, i.e. define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) := \frac{\int_{-\infty}^{x} \sigma(t)\sigma(\frac{1}{2} - t)\ dt}{\int_{-\infty}^{\infty} \sigma(t)\sigma(\frac{1}{2} - t)\ dt}$$

Now, set $\beta(x) := g(1 + x)g(1 - x)$ whose graph is



This essentially solves the problem: It only remains to set

$$f(x) := \prod_{i=1}^{n} \beta\left(\frac{x_i - a_i}{\varepsilon}\right).$$

We are now in a position to prove the actual statement of the Lemma: Let $\varepsilon > 0$ and $a_1, \ldots, a_l \in K$ such that $K \subseteq \bigcup_{i=1}^{l} C^n_{\varepsilon/2}(a_i) \subseteq \bigcup_{i=1}^{l} C^n_\varepsilon(a_i) \subseteq \mathbb{R}^n \setminus F$. For each a_i ($i \in \{1, \ldots, l\}$) pick β_i as above for the hypercube $C^n_{\varepsilon/2}(a_i)$ and set $f(x) = 1 - \prod_{i=1}^{l}(1 - \beta_i(x))$. This function has all the right properties: Firstly, it ranges betweeen zero and one because the β_i do. Secondly, for $x \in K$ there is some $i_0 \in \{1, \ldots, l\}$ such that $x \in C^n_{\varepsilon/2}(a_{i_0})$; hence $\beta_{i_0}(x) = 1$ whence $f(x) = 1$. Thirdly, if $x \in F$ then $x \notin C^n_\varepsilon(a_i)$ and hence $\beta_i(x) = 0$ for all $i \in \{1, \ldots, l\}$ and therefore $f(x) = 0$. This proves the assertion. ☐

A good account for analytical aspects of the theory as the one above is [dR56]. The next lemma states that the image of a derivation on a function is determined by the local behaviour of that function.

Lemma 2.10. Let N be a manifold and $p \in N$. Further, let $\delta \in \mathrm{Der}_p(C^\infty(N))$ and $g \in C^\infty(N)$ such that $g \equiv 1$ in a neighbourhood of p. Then $\delta(f) = \delta(fg)$ for all $f \in C^\infty(N)$.

Proof. By linearity of δ it suffices to show that $\delta(h) = 0$ where $h := f - fg$. Since h vanishes in a neighbourhood V of $p \in N$ there is some $\psi \in C^\infty(N)$ whose support is contained in V and which satisfies $\psi(p) \neq 0$: Indeed, let (U, φ) be a chart at p whose domain is contained in V. Let C be a hypercube centered at $\varphi(p)$ contained in $\varphi(U)$ and let $\beta \in C^\infty(\mathbb{R}^n)$ be a function adapted to C as in the previous lemma. Then set $\psi := \beta \circ \varphi$, naturally extended to the whole of N.

Then $h \cdot \psi = 0$ and hence $0 = \delta(h \cdot \psi) = h(p)\delta(\psi) + \delta(h)\psi(p)$. Since $h(p) = 0$ and $\psi(p) \neq 0$ this implies $\delta(h) = 0$ and hence $\delta(f) = \delta(fg)$. □

We now get to the C^∞-part of the argument without which Theorem 2.8 fails.

Lemma 2.11. Let $a \in \mathbb{R}^n$ and $U \subseteq \mathbb{R}^n$ a star-shaped neighbourhood of a. Further, let $f \in C^\infty(U)$. Then there are $g_1, \dots, g_n \in C^\infty(U)$ with

(i) $(g_1(a), \dots, g_n(a)) = \left(\frac{\partial f}{\partial x_1}(a), \dots, \frac{\partial f}{\partial x_n}(a)\right)$, and
(ii) $f(x) = f(a) + \sum_{i=1}^n (x_i - a_i) g_i(x)$

In a way, the above lemma formulates a factorization property: For instance, if $n = 1$, it asserts $f(x) - f(a) = (x - a)g(x)$ where g is again a C^∞-function. If f was only C^k for finite k then g would be at most C^{k-1}.

Proof. Since U is star-shaped with respect to a, we may write

$$f(x) = f(a) + \int_0^1 \frac{\partial f}{\partial t}(a + t(x - a))\ dt$$

Furthermore, by the chain rule

$$\frac{\partial f}{\partial t}(a + t(x - a)) = \sum_{i=1}^n (x_i - a_i) \frac{\partial f}{\partial x_i}(a + t(x - a))$$

and hence

$$f(x) = f(a) + \sum_{i=1}^n (x_i - a_i) \underbrace{\int_0^1 \frac{\partial f}{\partial x_i}(a + t(x - a))\ dt}_{=: g_i(x)}.$$

In particular, $g_i(a) = \int_0^1 (\partial f / \partial x_i)(a)\ dt = (\partial f / \partial x_i)(a)$. □

Despite this factorization property, C^∞-functions can be pretty "bad". In fact it is a result of Borel that given a sequence $(a_n)_n$ of real numbers there is a function $f \in C^\infty(\mathbb{R})$ with $f^{(n)}(0) = a_n$. This, however, is far from true for real analytic functions.

We are now in a position to prove Theorem 2.8. As announced, we translate the problem to one in $\mathbb{R}^n$ and solve it there.

Proof. (Theorem 2.8). Let (U, φ) be a chart at p. Pick $\varepsilon > 0$ such that $C^n_{2\varepsilon}(\varphi(p)) \subseteq \varphi(U)$. Now let $g \in C^\infty(\mathbb{R}^n)$ be as f in the proof of Lemma 2.9, i.e. g equals one on $C^n_{\varepsilon/2}(\varphi(p))$ and g equals zero outside $C^n_\varepsilon(\varphi(p))$.

For $f \in C^\infty(N)$, we have $\delta(f) = \delta(f \cdot (g \circ \varphi))$ by Lemma 2.10 where we agree that $g \circ \varphi$ is extended to $N \backslash U$ by zero. This in turn equals $\delta(((f \circ \varphi^{-1}) \cdot g) \circ \varphi)$ where now $(f \circ \varphi^{-1}) \cdot g$ is a smooth function on $\mathbb{R}^n$ with support in $C^n_\varepsilon(\varphi(p))$.

Now, for every $F \in C^\infty(C^n_{2\varepsilon}(\varphi(p)))$ define $\alpha(F) := \delta((F \cdot g) \circ \varphi)$ which is a derivation of $C^\infty(C^n_{2\varepsilon}(\varphi(p)))$ at $\varphi(p)$. For ease of notation, set $a := \varphi(p)$. Apply Lemma 2.11 to F to get functions $G_i \in C^\infty(C^n_{2\varepsilon}(a))$ such that

$$F(x) = F(a) + \sum_{i=1}^n (x_i - a_i) G_i(x).$$

Applying α yields

$$\begin{aligned}\alpha(F) &= \alpha(F(a)\cdot \mathbf{1}) + \sum_{i=1}^{n} \alpha(x \mapsto (x_i - a_i))G_i(a) + \sum_{i=1}^{n} 0\cdot \alpha(G_i) \\ &= \sum_{i=1}^{n} \underbrace{\alpha(x \mapsto (x_i - a_i))}_{c_i} \frac{\partial F}{\partial x_i}(a).\end{aligned}$$

We therefore have

$$\begin{aligned}\delta(f) = \alpha(f\circ\varphi^{-1}) &= \sum_{i=1}^{n} c_i \frac{\partial (f\circ\varphi^{-1})}{\partial x_i}\varphi(p) \\ &= D_{\varphi(p)}(f\circ\varphi^{-1})(v) = (D_p f)(D_{\varphi(p)}\varphi^{-1})(v)\end{aligned}$$

which is the assertion. □

Now, let N and M be manifolds and let $f : N \to M$ be a smooth map. Define $f^* : C^\infty(M) \to C^\infty(N)$ by $g \mapsto g\circ f$. This is an $\mathbb{R}$-algebra homomorphism. Given $p \in N$ we further define

$$f_* : \mathrm{Der}_p(C^\infty(N)) \to \mathrm{Der}_{f(p)}(C^\infty(M)),\ \delta \mapsto \delta\circ f^*.$$

The map f_* does indeed range in $\mathrm{Der}_p(C^\infty(M))$: Given $\delta \in \mathrm{Der}_p(C^\infty(N))$ and $g_1, g_2 \in C^\infty(M)$, we have

$$\begin{aligned}f_*\delta(g_1 g_2) &= \delta(f^*(g_1 g_2)) = \delta((f^* g_1)(f^* g_2)) = \delta((g_1\circ f)(g_2\circ f)) \\ &= (g_1\circ f)(p)\delta(g_2\circ f) + (g_2\circ f)(p)\delta(g_1\circ f) \\ &= g_1(f(p))f_*(\delta)(g_2) + g_2(f(p))f_*(\delta)(g_1).\end{aligned}$$

Also, the following diagram commutes:

$$\begin{array}{ccc} \mathrm{T}_p N & \xrightarrow{\ D_p f\ } & \mathrm{T}_{f(p)} M \\ \Big\downarrow{\scriptstyle\cong} & & \Big\downarrow{\scriptstyle\cong} \\ \mathrm{Der}_p\, C^\infty(N) & \xrightarrow[\ f_*\ ]{} & \mathrm{Der}_{f(p)}\, C^\infty(M) \end{array}$$

In particular, the derivative of f can be defined without ever writing down an actual derivative in $\mathbb{R}^n$. This algebraization of the derivative is used a lot in algebraic geometry. It is also very useful when working with vector fields as we shall see later. However, both versions of the tangent space and the derivative are important.

2.3. **The Tangent Bundle.** In this section, we want to show how one can organize the set of tangent vectors of a manifold M into a manifold $\mathrm{T}M$ on its own. For instance, this is essential to define smooth vector fields and plays a major rule in Whitney's embedding theorem which we will see later. As a set, we have

$$\mathrm{T}M = \bigcup_{x\in M} \{x\} \times \mathrm{T}_x M = \{(x,v) \mid x \in M,\ v \in \mathrm{T}_x M\}.$$

The tangent bundle $\mathrm{T}M$ of a manifold M comes with the natural projection map $\pi : \mathrm{T}M \to M,\ (x,v) \mapsto x$. For any subset $U \subseteq M$ we define

$$\mathrm{T}U := \pi^{-1}(U) = \{(x,v) \mid x \in U,\ v \in \mathrm{T}_x M\}$$

and use these subsets of $\mathrm{T}M$ to define a topology as follows: Let (U,φ) be any chart of M. Then $D\varphi : \mathrm{T}U \to \varphi(U)\times\mathbb{R}^m$ given by $(x,v) \mapsto (\varphi(x), D_x\varphi(v))$ is a bijection. We now declare a subset $E \subseteq \mathrm{T}M$ to be open if and only if for every chart (U,φ) of M the set $D\varphi(E\cap \mathrm{T}U) \subseteq \varphi(U)\times\mathbb{R}^m$ is open.

Lemma 2.12. Retain the above notation. Then $\mathrm{T}M$ is a Hausdorff and second countable topological space. Also, given a chart (U, φ) of M the map $D\varphi$ from $\mathrm{T}U$ to $\varphi(U) \times \mathbb{R}^m$ is a homeomorphism, and $\pi : \mathrm{T}M \to M$ is continuous and open.

Proof. First of all, we show that the set

$$\{E \subseteq \mathrm{T}M \mid \forall (U, \varphi) \text{ chart of } M: \; D\varphi(E \cap \mathrm{T}U) \subseteq \varphi(U) \times \mathbb{R}^m \text{ is open}\}$$

defines a topology on $\mathrm{T}M$. Note that for a chart (U, φ) of M the subset $\mathrm{T}U \subseteq \mathrm{T}M$ is open. Hence so is $\mathrm{T}M$ since $\mathrm{T}M \cap \mathrm{T}U = \mathrm{T}U$. Also, the empty set is open. As to finite intersections, suppose that $E_1, \ldots, E_n$ are open subsets of $\mathrm{T}M$. Then

$$\bigcap_{i=1}^{n} E_i \cap \mathrm{T}U = \bigcap_{i=1}^{n} (E_i \cap \mathrm{T}U)$$

is open. Also, if $(E_\alpha)_{\alpha \in A}$ is a family of open subsets of $\mathrm{T}M$ then

$$\bigcup_{\alpha \in A} E_\alpha \cap \mathrm{T}U = \bigcup_{\alpha \in A} (E_\alpha \cap \mathrm{T}U)$$

is open. The definition also readily implies that $D\varphi : \mathrm{T}U \to \varphi(U) \times \mathbb{R}^m$ is a homeomorphism for every chart (U, φ) of M.

With the above topology, $\mathrm{T}M$ is a Hausdorff space: Indeed, let $(p_1, v_1) \neq (p_2, v_2)$ be distinct points in $\mathrm{T}M$. If $p_1 \neq p_2$, choose charts (U_1, φ_1) and (U_2, φ_2) at p_1 and p_2 respectively such that $U_1 \cap U_2 = \emptyset$. Then $\mathrm{T}U_1$ and $\mathrm{T}U_2$ are non-intersecting open neighbourhoods of (p_1, v_1) and (p_2, v_2) respectively. If $p_1 = p_2$ then $v_1 \neq v_2$. Hence, if (U, φ) is a chart at $p := p_1 = p_2$ then $(p, v_1), (p, v_2) \in \mathrm{T}U \cong \varphi(U) \times \mathbb{R}^m$ and we may use the fact that $\mathbb{R}^m$ is a Hausdorff space to separate (p_1, v_1) and (p_2, v_2). Second countability is left to the reader as an exercise.

It remains to show that $\pi : \mathrm{T}M \to M$ is continuous and open. Given an atlas $\mathcal{A} = \{(U_\alpha, \varphi_\alpha \mid \alpha \in A\}$ of M we have $M = \bigcup_{\alpha \in A} U_\alpha$ and it suffices to show that $\pi|_{U_\alpha} : \mathrm{T}U_\alpha \to U_\alpha$ is continuous and open for every $\alpha \in A$. This follows from the fact the map pr_1 in the following diagram is continous and open,

$$\begin{array}{ccc} \mathrm{T}U_\alpha & \xrightarrow[\cong]{D\varphi_\alpha} & \varphi_\alpha(U_\alpha) \times \mathbb{R}^m \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \mathrm{pr}_1} \\ U_\alpha & \xrightarrow[\varphi_\alpha]{\cong} & \varphi_\alpha(U_\alpha), \end{array}$$

because $D\varphi_\alpha$ and φ_α are homeomorphisms. □

The proof of Lemma 2.12 in fact shows that if $\mathcal{A} = \{(U_\alpha, \varphi_\alpha \mid \alpha \in A\}$ is an atlas on M then $\{(\mathrm{T}U_\alpha, D\varphi_\alpha) \mid \alpha \in A\}$ is a C^0-atlas on $\mathrm{T}M$. This atlas is in fact smooth.

Lemma 2.13. Retain the above notation. The atlas $\{(\mathrm{T}U_\alpha, D\varphi_\alpha) \mid \alpha \in A\}$ on $\mathrm{T}M$ is smooth.

Proof. We determine the coordinate transformations. Let $\alpha, \beta \in A$. Then we have $\mathrm{T}U_\alpha \cap \mathrm{T}U_\beta = T(U_\alpha \cap U_\beta)$ and

$$\begin{array}{ccc} & T(U_\alpha \cap U_\beta) & \\ {\scriptstyle D\varphi_\alpha} \swarrow & & \searrow {\scriptstyle D\varphi_\beta} \\ \varphi_\alpha(U_\alpha \cap U_\beta) \times \mathbb{R}^m & \xrightarrow[(\varphi_\beta \circ \varphi_\alpha^{-1}, D(\varphi_\beta \circ \varphi_\alpha^{-1}))]{} & \varphi_\beta(U_\alpha \cap U_\beta) \times \mathbb{R}^m . \end{array}$$

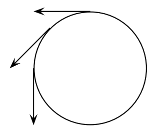
In particular, the coordinate transformations are smooth. □

As a consequence of the proof of Lemma 2.12 we also record that $\pi : \mathrm{T}M \to M$ has constant rank m as in local coordinates it is given by a projection in Euclidean space.

The smooth structure on $\mathrm{T}M$ allows to define smooth vector fields.

Definition 2.14. Let M be a manifold. A (continuous, C^k, smooth) *tangent vector field* on M is a (continuous, C^k, smooth) map $X : M \to \mathrm{T}M$ with $\pi \circ X = \mathrm{id}_M$.

It is an important question whether or not a manifold admits a nowhere vanishing continuous or smooth vector field as this potentially allows us to distinguish between given manifolds. For instance, S^1 clearly admits such a vector field. We shall see later that in fact any smooth vector field on S^2 has to have a zero somewhere. As a third example, note that $\mathrm{SO}(3)$ admits a nowhere vanishing smooth vector field: Simply choose a non-zero tangent vector v at $\mathrm{Id} \in \mathrm{SO}(3)$ and translate it around using the derivatives of the smooth maps $L_g : \mathrm{SO}(3) \to \mathrm{SO}(3),\ x \mapsto gx$ where $g \in \mathrm{SO}(3)$. The reader is invited to think about the case of S^3.



2.4. **Whitney's Embedding Theorem.** In this section we prove that every (compact) manifold can be embedded into some Euclidean space. The strong version of this Theorem due to Whitney reads as follows.

Theorem. Let M be an m-manifold. Then M embeds into $\mathbb{R}^{2m}$.

In full generality, the dimension of the Euclidean target space cannot be reduced any further: For instance, the two-dimensional manifold $\mathbb{P}^2\,\mathbb{R}$ cannot be embedded into $\mathbb{R}^3$ for orientability reasons. We shall prove the following version of the above theorem.

Theorem 2.15. Let M be a compact m-manifold. Then M embeds into $\mathbb{R}^{2m+1}$.

A byproduct of the proof of Theorem 2.15 is that in fact M can be immersed into $\mathbb{R}^{2m}$. However, it requires a new idea to get rid of potential double points.

Proof. The first step of the proof constitutes in constructing an embedding of M into a Euclidean space of some large dimension. To this end, recall Proposition 1.42 by which it suffices to construct an immersion of M into Euclidean space which is a homeomorphism onto its image. We then work on reducing the dimension using Sard's Theorem 1.43.

Given $p \in M$, let (U_p, φ_p) be a chart at p with $\varphi_p(p) = 0$ and pick $\varepsilon_p > 0$ such that $C^m_{3\varepsilon_p}(0) \subseteq \varphi_p(U_p)$. Further, given any $\varepsilon > 0$, let $\psi_\varepsilon : \mathbb{R}^m \to [0,1]$ be a smooth function with $\psi_\varepsilon^{-1}(1) = \overline{C^m_\varepsilon(0)}$ and $\psi_\varepsilon = 0$ outside of $C^m_{2\varepsilon}(0)$. Now define

$$f_p : M \to \mathbb{R},\ x \mapsto \begin{cases} \psi_{\varepsilon_p} \circ \varphi_p(x) & x \in U_p \\ 0 & x \notin U_p \end{cases}$$

which is smooth for every $p \in M$. Using f_p we further define

$$F_p : M \to \mathbb{R}^m,\ x \mapsto \begin{cases} f_p(x) \cdot \varphi_p(x) & x \in U_p \\ 0 & x \notin U_p \end{cases}.$$

Let $V_p := \varphi_p^{-1}(C^m_{\varepsilon_p}(0)) \subseteq U_p$ which is open and contains $p \in M$. Since M is compact, there are finitely many points $p_1, \dots, p_l \in M$ such that $M = \bigcup_{i=1}^{l} V_{p_i}$. Then the following map is the announced injective immersion:

$$\Psi : M \to \mathbb{R}^{ml+l},\ x \mapsto (F_{p_1}(x), \dots, F_{p_l}(x), f_{p_1}(x), \dots, f_{p_l}(x)).$$

Indeed, Ψ is of rank m everywhere: Let $x \in M$ and $i \in \{1, \dots, l\}$ such that $x \in V_{p_i}$. Consider

$$\Psi : M \to \mathbb{R}^{ml+l} = \underbrace{\mathbb{R}^m \times \cdots \times \mathbb{R}^m \times \cdots \times \mathbb{R}^m}_{l} \times \mathbb{R} \times \cdots \times \mathbb{R}$$

and denote by p_i the projection of $\mathbb{R}^{ml+l}$ onto the i-th $\mathbb{R}^m$-factor. Then $p_i \circ \Psi$ agrees with φ_{p_i} on a neighbourhood of x. As a consequence Ψ has (maximal) rank m at $x \in M$ since φ_{p_i} does. Note that in order to get an immersion from M to a Euclidean space we could have dropped the coordinates $f_{p_1}, \dots, f_{p_l}$ from the definition of Ψ. However, we need them to ensure injectivity: Let $x, y \in M$ and $i \in \{1, \dots, l\}$ be such that $x \in \overline{V}_{p_i} = \varphi_{p_i}^{-1}(\overline{C^m_{\varepsilon_p}(0)})$. First suppose that $y \in \overline{V}_{p_i}$ as well and assume that $\Psi(x) = \Psi(y)$. Since F_{p_i} and φ_{p_i} agree on $\overline{V}_{p_i}$ we conclude that $\varphi_{p_i}(x) = \varphi_{p_i}(y)$ and hence $x = y$. Now suppose that $y \notin \overline{V}_{p_i}$. Then $f_{p_i}(y) < 1$. But since $f_{p_i}(x) = 1$ this implies that $\Psi(x) \neq \Psi(y)$.

The strong version of Whitney's embedding theorem requires an additional idea in this step to deal with non-compactness.

As a second step, we now reduce the dimension of the target Euclidean space utilizing Sard's Theorem 1.43. We have an embedding of M into some large $\mathbb{R}^n$ and consider $\mathrm{T}M$ as a regular submanifold of $\mathrm{T}\,\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$, and in fact

$$\sigma(M) = \{(p, v) \in \mathrm{T}M \mid \|v\| = 1\}$$

as a $(2m-1)$-dimensional regular submanifold of $\mathbb{R}^n \times \mathbb{R}^n$. We now identify $\mathbb{R}^{n-1}$ as $\{x \in \mathbb{R}^n \mid x_n = 0\}$ and let $v \in \mathbb{R}^n \setminus \mathbb{R}^{n-1}$ with $\|v\| = 1$. Further, let $p_v : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be the projection parallel to v given by the decomposition $\mathbb{R}^n = \mathbb{R}^{n-1} \oplus \mathbb{R}\, v$. Now consider the map $f : \sigma(M) \to \mathbb{S}^{n-1}$ given by $(p, w) \mapsto w$ and observe that $p_v|_M$ is an immersion if and only if $v \notin f(\sigma M)$. If $\dim(\sigma(M)) = 2m-1 < n-1 = \dim S^{n-1}$, then by Sard's Theorem $f(\sigma M)$ has measure zero in S^{n-1}. Hence there is $v \in S^{n-1}$ with $v \notin f(\sigma M)$ for which consequently $p_v : M \to \mathbb{R}^{n-1}$ is an immersion. Repeating the argument, one eventually gets an immersion of M to $\mathbb{R}^{2m}$.

As a third step, we adress the injectivity issue. In doing so, we loose one dimension. The strong version of Whitney's Theorem avoids with using a deeper understanding about how to get rid of potentially introduced double points. Consider

$$g : (M \times M) \backslash \Delta(M) \to S^{n-1},\ (x, y) \mapsto \frac{x - y}{\|x - y\|}$$

which, using the immersion above, is smooth as the restriction of a smooth map from $\mathbb{R}^n \times \mathbb{R}^n$. Observe that $p_v|_M$ is injective if and only if $v \notin \mathrm{im}(g)$: We are only intersted in the following direction: Suppose that $p_v|_M$ is not injective. Then there are $x, y \in M$ with $p_v(x) = p_v(y)$ and hence $p_v(x - y) = 0$, i.e. $x - y \in \mathbb{R}\, v$. Therefore $g(x, y) \in \{\pm v\}$ and hence $v \in \mathrm{im}(g)$. By Sard's Theorem, the image of g has measure zero in S^{n-1} if $2m < n - 1$. Putting everything together, we have that if $2m < n - 1$, the set $\mathrm{im}(f) \cup \mathrm{im}(g)$ has measure zero in S^{n-1}. Therefore we can find $v \notin \mathrm{im}(f) \cup \mathrm{im}(g)$ for which p_v is an injective immersion and hence an embedding by Proposition 1.42 since M is compact. □

Whitney's Theorem and Sard's Theorem can be applied to obtain considerable information about a given manifold. For instance, suppose that M is a compact manifold and that $f : M \to \mathbb{R}$ is a smooth map all of whose values are regular. Then $f^{-1}(y)$ for $y \in \mathrm{im}(f)$ is a regular submanifold of M of lower dimension and in fact M is built up from such lower-dimensional submanifolds in a certain way. However, a function $f : M \to \mathbb{R}$ does not exist as for instance a point $x \in M$ at which f assumes its maximum is critical. However, one can still insist on using this method and try to obtain more precise information about these critical values using the second-derivative of f. A Morse function is one all of whose second derivatives are non-degenerate. Combining Whitney's and Sard's Theorem one can show the existence of Morse functions which can be used to analyze the structure of M. For instance, this yields a classification of all compact surfaces.

As a second remark, we state that although the dimension of the Euclidean space in the strong version of Whitney's theorem is optimal for some m, for instance powers of two, for which projective space realizes the worst case, it is not in general. However, the general relation is a rather complicated one.

2.5. **The Cotangent Bundle.** In this section we introduce the cotangent bundle of a manifold in analogy to its tangent bundle and it turns out to be even more important. First, we fix some notation. Given an $\mathbb{R}$-vector space V, let $V^* = \mathrm{Lin}(V, \mathbb{R})$ denote its dual space. If $T : V \to W$ is a linear map of $\mathbb{R}$-vector spaces V and W, then $T^* : W^* \to V^*$ denotes its adjoint map, given by $(T^*\lambda)(v) = \lambda(Tv)$. Now, let M be a smooth manifold of dimension m. As in the case of the tangent bundle, cf. Section 2.3 we define a smooth structure on $\mathrm{T}^*M = \bigcup_{x\in M}\{x\} \times \mathrm{T}_x M^*$. This leads to the cotangent bundle $\mathrm{T}^*M \xrightarrow{\pi} M$ with smooth, submersive π. The topology and the smooth structure on T^*M are defined using charts (U, φ) on M: Let $\mathrm{T}^*U = \pi^{-1}(U)$. Then we have the map

$$\mathrm{T}^*U \to \varphi(U) \times (\mathbb{R}^m)^*,\ (x, \lambda) \mapsto (\varphi(x), (D_{\varphi(x)}\varphi^{-1})^*(\lambda)).$$

From here on, one proceeds as in Section 2.3.

Definition 2.16. Let M be a manifold. A (continuous, C^k, smooth) *differential* 1-*form on* M is a (continuous, C^k, smooth) map $\omega : M \to \mathrm{T}^*M$ such that $\pi\circ\omega = \mathrm{id}_M$.

In a sense, differential 1-forms are more natural then vector fields: Let $\Omega^1(M)$ be the vector space of smooth 1-forms on M. The derivative of smooth functions gives rise to a natural map

$$d : C^\infty(M) \to \Omega^1(M),\ f \mapsto (df : x \mapsto D_x f)$$

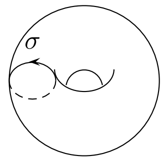
where by "naturality" we mean the following: Recall that a smooth map $\psi : N \to M$ between manifolds induces an algebra homomorphism $\psi^* : C^\infty(M) \to C^\infty(N)$ and a map $\psi^* : \Omega^1(M) \to \Omega^1(N)$ given by $(\psi^*\omega)_x(v) = \omega_{\psi(x)} D_x\psi(v)$. The construction d is now a natural transformation between the functors $C^\infty(-)$ and $\Omega^1(-)$, i.e. the following diagram commutes.

$$\begin{array}{ccc} C^\infty(M) & \xrightarrow{d_M} & \Omega^1(M) \\ {\scriptstyle\psi^*}\downarrow & & \downarrow{\scriptstyle\psi^*} \\ C^\infty(N) & \xrightarrow[d_N]{} & \Omega^1(N). \end{array}$$

Also, note that $\Omega^1(M)$ is not only a vector space but in fact a $C^\infty(M)$-module with multiplicative structure given by $(f\omega)_x := f(x)\omega_x$.

We now start to generalize calculus on $\mathbb{R}^m$ to smooth manifolds. For instance, what kind of object can be integrated over a manifold? Looking at the case of $\mathbb{R}^m$ it seems like the answer should be "functions" and indeed one could try to define an integral of a function over a manifold by patching together integrals in chart codomains but this would not be well-defined as the integral in $\mathbb{R}^m$ is not invariant under diffeomorphisms. The right objects to integrate will turn out to be k-forms.

We can already see that a 1-form ω could be integrated over a one-dimensional regular submanifold $\sigma : [0,1] \to M$ of a manifold by defining $\int_\sigma \omega := \int_0^1 \omega_{c(t)}\dot{c}(t)\ dt$.

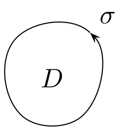


2.6. **Differential 1-forms in Local Coordinates.** In order to give a meaning to the expressions that occur in e.g. Green's formula in the introduction we now express 1-forms in local coordinates. To this end, let M be a smooth m-manifold and $\omega \in \Omega^1(M)$ a smooth 1-form. Further, let (U, φ) be any coordinate chart of M. On $\mathbb{R}^m$ we have the coordinate functions $\pi_i : \mathbb{R}^m \to \mathbb{R}$ given by $x = (x_1, \ldots, x_m)^T \mapsto x_i$. These functions give rise to the 1-forms $d\pi_1, \ldots, d\pi_m$ on $\mathbb{R}^m$ which at every point $x \in \mathbb{R}^m$ give a basis of $(\mathrm{T}_x\,\mathbb{R}^m)^* = (\mathbb{R}^m)^*$. In fact, for $x \in \mathbb{R}^m$ and $v \in \mathbb{R}^m$ we have $(d\pi_i)_x v = v_i$ which is essentially due to the fact that the differential of a linear map is the map itself. Now, since $\varphi : U \to \varphi(U)$ is a diffeomorphism onto its image, the pullback forms $\varphi^*(d\pi_1), \ldots, \varphi^*(d\pi_m)$ form a basis of $\mathrm{T}_p M^*$ at every point $p \in U$.

To shorten the notation, we shall simply write $dx_i := \varphi^*(d\pi_i) \in \Omega^1(U)$. We then have $\omega_x = \sum_{i=1}^n a_i(x)(dx_i)_x$ and the functions $a_i : U \to \mathbb{R}$ are smooth. We will also write $\omega = \sum_{i=1}^n a_i\ dx_i$ in the $C^\infty(U)$-module $\Omega^1(U)$. Furthermore, if $V \subseteq \mathbb{R}^m$ is open we will also consider V as a smooth manifold with the single chart (V, id) and use the notation $dx_i \in \Omega^1(V)$ for the 1-form obtained by using the chart (V, id).

3. Differential Forms and Integration on Manifolds

We have seen above that 1-forms can be integrated over one-dimensional manifolds in an invariant way. In this section we introduce k-forms which constitute the natural objects to be integrated over k-dimensional manifolds. As a motivation, consider again Green's formula from the introduction:



$$\int_\sigma \omega = \oint_{\partial D} P\ dx + Q\ dy = \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)\ dx\ dy.$$

If $\omega \in \Omega^1(\mathbb{R}^2)$ then $\omega = P\ dx + Q\ dy$ for smooth functions $P, Q : \mathbb{R}^2 \to \mathbb{R}$ by the above. Note that the expressions dx and dy now have a precise meaning, namely they are 1-forms on $\mathbb{R}^2$, which is typically vague in calculus courses. The first two terms in Green's formula are now well-defined.

The remainder of this section in particular identifies $(\partial Q/\partial x - \partial P/\partial y)\ dx\ dy$ as a 2-form, the *exterior derivative* $d\omega$ of ω, and defines intregration on manifolds M with boundary of such forms. The pinnacle will be Stokes' Theorem

$$\int_{\partial M} \omega = \int_M d\omega$$

which is arguably one of the most wonderful formulas in mathematics.

3.1. **Alternating Forms on Vector Spaces.** To implement the above program we require several multilinear algebra notions. Let V be a finite-dimensional real vector space (neither finite-dimensionality nor real coefficients are required everywhere but we shall not worry about these things here).

Definition 3.1. Retain the above notation. A *multilinear k-form on V* is a function $\mu : V \times \cdots \times V \to \mathbb{R}$ which is linear in each argument.

Lemma 3.2. Retain the above notation and let μ be a multilinear k-form on V. Then the following are equivalent.

(i) The form μ is zero whenever two arguments coincide.
(ii) The form μ changes its sign whenever two arguments are interchanged.
(iii) For all $v_1, \ldots, v_k \in V$ and $\sigma \in S_k$: $\mu(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = \operatorname{sign}(\sigma)\mu(v_1, \ldots, v_k)$.

Based on Lemma 3.2 we make the following definition.

Definition 3.3. Retain the above notation. A *multilinear, alternating k-form on V* is a multilinear form on V satisfying one of the equivalent properties of Lemma 3.2.

Proof. (Lemma 3.2). We show that (i) implies (ii): Let $v_1, \ldots, v_k \in V$. Then we have for all $1 \le i < j \le n$:

$$\begin{aligned} 0 &= \mu(v_1, \ldots, v_{i-1}, v_i + v_j, v_{i+1}, \ldots, v_{j-1}, v_i + v_j, v_{j+1}, \ldots, v_n) \\ &= \mu(\ldots, v_i, \ldots, v_i, \ldots) + \mu(\ldots, v_i, \ldots, v_j, \ldots) + \\ &\qquad + \mu(\ldots, v_j, \ldots, v_i, \ldots) + \mu(\ldots, v_j, \ldots, v_j, \ldots) \\ &= \mu(\ldots, v_i, \ldots, v_j, \ldots) + \mu(\ldots, v_j, \ldots, v_i, \ldots). \end{aligned}$$

Hence the assertion. Also, (ii) implies (iii): Rephrasing (ii), we have

$$\mu(v_{\tau(1)}, \ldots, v_{\tau(k)}) = (-1) \cdot \mu(v_1, \ldots, v_k)$$

for every transposition $\tau \in S_k$. Now use the fact that every permutation $\sigma \in S_k$ can be written as a product of transpositions and that $\operatorname{sign} : S_k \to \{\pm 1\}$ is a group homomorphism. The implications (iii)⇒(ii) and (ii)⇒(i) are immediate. □

We now organize the alternating forms on a vector space into a graded algebra: First of all, given $k \in \mathbb{N}_0$, let $\Lambda^k(V^*)$ denote the vector space of alternating k-forms on V where we define $\Lambda^0(V^*) := \mathbb{R}$. In particular, we have $\Lambda^1(V^*) = V^*$. Now consider the graded vector space

$$\Lambda^*(V^*) = \bigoplus_{k \ge 0} \Lambda^k(V^*),$$

termed the *exterior algebra* of V^* because it admits the following multiplication, termed *wedge product*: Let $\alpha \in \Lambda^p(V^*)$ and $\beta \in \Lambda^q(V^*)$. Define

$$\begin{aligned} (\alpha \wedge \beta)(v_1, \ldots, v_{p+q}) &:= \frac{1}{p!q!} \sum_{\sigma \in S_{p+q}} \operatorname{sign}(\sigma)\alpha(v_{\sigma(1)}, \ldots, v_{\sigma(p)})\beta(v_{\sigma(p+1)}, \ldots, v_{\sigma(p+q)}) \\ &= \sum_{\substack{\sigma \in S_{p+q} \\ \sigma(1) < \cdots < \sigma(p) \\ \sigma(p+1) < \cdots < \sigma(p+q)}} \operatorname{sign}(\sigma)\alpha(v_{\sigma(1)}, \ldots, v_{\sigma(p)})\beta(v_{\sigma(p+1)}, \ldots, v_{\sigma(p+q)}) \end{aligned}$$

where equality is left as an exercise. The special kind of permutation of $\{1, \ldots, p+q\}$ that occurs in the second expression is called (p, q)-shuffle for apparent reasons.

Proposition 3.4. Retain the above notation. Then $\Lambda^*(V^*)$ is an associative, graded commutative $\mathbb{R}$-algebra.

Before we proceed to the proof of Proposition 3.4 several remarks are in order.

Remark 3.5. Let $\alpha \in \Lambda^p(V^*)$ and $\beta \in \Lambda^q(V^*)$.

(i) If $p = 0$ then $\alpha \wedge \beta = \alpha \cdot \beta$.
(ii) The term *graded-commutative* means that $\beta \wedge \alpha = (-1)^{pq} \alpha \wedge \beta$. In particular, even order forms commute with any other form.

(iii) The definition of the wedge product may be motivated as follows: Given an multilinear k-form μ on V there is a natural alternating k-form A_μ on V associated to μ, namely

$$A_\mu(v_1,\dots,v_k) := \frac{1}{k!}\sum_{\sigma\in S_k}\operatorname{sign}(\sigma)\mu(v_{\sigma(1)},\dots,v_{\sigma(k)}).$$

It is immediate that A_μ is multilinear. To see that it is also alternating we let $v_1,\dots,v_k\in V$, $\tau\in S_k$ and compute

$$\begin{aligned}A_\mu(v_{\tau(1)},\dots,v_{\tau(k)}) &= \frac{1}{k!}\sum_{\sigma}\operatorname{sign}(\sigma)\operatorname{sign}(\sigma)\mu(v_{\tau\sigma(1)},\dots,v_{\tau\sigma(k)})\\ &= \frac{1}{k!}\sum_{\sigma\in S_k}\operatorname{sign}(\tau^{-1}\sigma)\mu(v_{\sigma(1)},\dots,v_{\sigma(k)})\\ &= \operatorname{sign}(\tau)\frac{1}{k!}\sum_{\sigma\in S_k}\operatorname{sign}(\sigma)\mu(v_{\sigma(1)},\dots,v_{\sigma(k)})\\ &= \operatorname{sign}(\tau)A_\mu(v_1,\dots,v_k)\end{aligned}$$

Now consider $\mu(v_1,\dots,v_{p+a}) := \alpha(v_1,\dots,v_p)\beta(v_{p+1},\dots,v_{p+q})$ which is multilinear. Then $\alpha\wedge\beta = (p+q)!/(p!q!)A_\mu$.

(iv) So far, we have only defined the multiplication within $\Lambda^*(V^*)$ on pairs of p and q-forms. However, we may simply extend this definition linearly to the whole of $\Lambda^*(V^*)$ realizing distributivity: Suppose $\alpha,\beta\in\Lambda^*(V^*)$ are given by $\alpha=\sum_i\alpha_i$ and $\beta=\sum_j\beta_j$ where $\alpha_i\in\Lambda^i(V^*)$ and $\beta_j\in\Lambda^j(V^*)$ then $\alpha\wedge\beta=\sum_{i,j}\alpha_i\wedge\beta_j$. However, mixed products are rare in practice.

Proof. (Proposition 3.4). First consider associativity: Let $\alpha\in\Lambda^p(V^*)$, $\beta\in\Lambda^q(V^*)$, $\gamma\in\Lambda^r(V^*)$ and $v_1,\dots,v_{p+q+r}\in V$. Then $((\alpha\wedge\beta)\wedge\gamma)(v_1,\dots,v_{p+q},\dots,v_{p+q+r})$ equals

$$\frac{1}{(p+q)!r!}\sum_{\sigma\in S_{p+q+r}}\operatorname{sign}(\sigma)(\alpha\wedge\beta)(v_{\sigma(1)},\dots,v_{\sigma(p+q)})\gamma(v_{\sigma(p+q+1)},\dots,v_{\sigma(p+q+r)})$$

which in turn equals

$$\begin{aligned}\frac{1}{(p+q)!p!q!r!}\sum_{\sigma\in S_{p+q+r}}\sum_{\tau\in S_{p+q}}\operatorname{sign}(\sigma)\operatorname{sign}(\tau)\,&\alpha(v_{\sigma\tau(1)},\dots,v_{\sigma\tau(p)})\cdot\\ &\cdot\beta(v_{\sigma\tau(p+1)},\dots,v_{\sigma\tau(p+q)})\cdot\\ &\cdot\gamma(v_{\sigma(p+q+1)},\dots,v_{\sigma(p+q+r)}).\end{aligned}$$

Now, to every $\tau\in S_{p+q}$ we associate $\widehat{\tau}\in S_{p+q+r}$ by fixing $p+q+1,\dots,p+q+r$:

$$\widehat{\tau}(j) := \begin{cases}\tau(j) & j\in\{1,\dots,p+q\}\\ j & j\in\{p+q+1,\dots,p+q+r\}\end{cases}.$$

Then the above expression can be rewritten as

$$\begin{aligned}\frac{1}{(p+q)!p!q!r!}\sum_{\sigma\in S_{p+q+r}}\sum_{\tau\in S_{p+q}}\operatorname{sign}(\sigma\hat\tau)\,&\alpha(v_{\sigma\hat\tau(1)},\dots,v_{\sigma\hat\tau(p)})\cdot\\ &\cdot\beta(v_{\sigma\hat\tau(p+1)},\dots,v_{\sigma\hat\tau(p+q)})\cdot\\ &\cdot\gamma(v_{\sigma\hat\tau(p+q+1)},\dots,v_{\sigma\hat\tau(p+q+r)}).\end{aligned}$$

Since the map $S_{p+q+r}\to S_{p+q+r}$, $\sigma\mapsto\sigma\hat\tau$ is a bijection we may continue with

$$\begin{aligned}\frac{1}{(p+q)!p!q!r!}\sum_{\tau\in S_{p+q}}\sum_{\sigma\in S_{p+q+r}}\operatorname{sign}(\sigma)\,&\alpha(v_{\sigma(1)},\dots,v_{\sigma(p)})\cdot\\ &\cdot\beta(v_{\sigma(p+1)},\dots,v_{\sigma(p+q)})\cdot\\ &\cdot\gamma(v_{\sigma(p+q+1)},\dots,v_{\sigma(p+q+r)}).\end{aligned}$$

which equals

$$\frac{1}{p!q!r!}\sum_{\sigma\in S_{p+q+r}}\operatorname{sign}(\sigma)\alpha(v_{\sigma(1)},\dots,v_{\sigma(p)})\,\beta(v_{\sigma(p+1)},\dots,v_{\sigma(p+q)})\cdot \\ \cdot\gamma(v_{\sigma(p+q+1)},\dots,v_{\sigma(p+q+r)}).$$

Computing $\alpha\wedge(\beta\wedge\gamma)$ in a similar way yields the same. This proves associativity.

We now turn to proving that $\Lambda^*(V^*)$ is graded commutative: Let $\alpha\in\Lambda^p(V^*)$, $\beta\in\Lambda^q(V^*)$ and $v_1,\dots,v_{p+q}\in V$. Then

$$(\alpha\wedge\beta)(v_1,\dots,v_{p+q})=\frac{1}{p!q!}\sum_{\sigma\in S_{p+q}}\operatorname{sign}(\sigma)\alpha(v_{\sigma(1)},\dots,v_{\sigma(p)})\beta(v_{\sigma(p+1)},\dots,v_{\sigma(p+q)})$$

Again, we use that the map $S_{p+q}\to S_{p+q},\ \sigma\mapsto\sigma\tau$ is a bijection for any $\tau\in S_{p+q}$. Applying this to

$$\tau:=\begin{pmatrix}1&\cdots&p&p+1&\cdots&p+q\\ q+1&\cdots&q+p&1&\cdots&q\end{pmatrix}\in S_{p+q}$$

which has sign $(-1)^{pq}$ yields

$$\frac{1}{p!q!}\sum_{\sigma\in S_{p+q}}\operatorname{sign}(\sigma\tau)\alpha(v_{\sigma\tau(1)},\dots,v_{\sigma\tau(p)})\beta(v_{\sigma\tau(p+1)},\dots,v_{\sigma\tau(p+q)})$$
$$=(-1)^{pq}\frac{1}{p!q!}\sum_{\sigma\in S_{p+q}}\operatorname{sign}(\sigma)\beta(v_{\sigma(1)},\dots,v_{\sigma(q)})\alpha(v_{\sigma(q+1)},\dots,v_{\sigma(q+p)})$$

which is $(-1)^{pq}(\beta\wedge\alpha)(v_1,\dots,v_{p+q})$. □

In order to get an impression of how to use graded commutativity, consider the following. Let V be a real three-dimensional vector space with basis (e_1,e_2,e_3) and let (e_1^*,e_2^*,e_3^*) be the associated dual basis of $V^*=\Lambda^1(V^*)$. Given $a_i,b_i\in\mathbb{R}$ ($i\in\{1,2,3\}$) we have the 1-forms $a_1e_1^*+a_2e_2^*+a_3e_3^*$ and $b_1e_1^*+b_2e_2^*+b_3e_3^*$. To compute their wedge product, which is a 2-form, note that for two $f,g\in V^*$ we have $f\wedge f=0$ since $f\wedge f=(-1)^{1\cdot 1}f\wedge f$ and $f\wedge g=-g\wedge f$ by graded commutativity. Therefore $(a_1e_1^*+a_2e_2^*+a_3e_3^*)\wedge(b_1e_1^*+b_2e_2^*+b_3e_3^*)$ equals

$$a_1b_2e_1^*\wedge e_2^*+a_1b_3e_1^*\wedge e_3^*+a_2b_1e_2^*\wedge e_1^*+a_2b_3e_2^*\wedge e_3^*+a_3b_1e_3^*\wedge e_1^*+a_3b_2e_3^*\wedge e_2^*$$
$$=(a_1b_2-a_2b_1)e_1^*\wedge e_2^*+(a_1b_3-a_3b_1)e_1^*\wedge e_3^*+(a_2b_3-b_2a_3)e_2^*\wedge e_3^*.$$

In general, the evaluation of a k-form on a k-tuple of vectors is a determinant.

Proposition 3.6. Let V be a finite-dimensional real vector space. Furthermore, let $\phi_1,\dots,\phi_k\in V^*$ and $v_1,\dots,v_k\in V$. Then $(\phi_1\wedge\dots\wedge\phi_k)(v_1,\dots,v_k)=\det(\phi_i(v_j))_{i,j}$.

Proof. We argue by induction on k. If $k=1$, then $\phi(v)=\det(\phi(v))$. Now, assume $k\geq 2$. We expand the determinant $\det(\phi_i(v_j))_{i,j=1}^k$ along the last row as

$$\sum_{j=1}^k(-1)^{k-1}(-1)^{j-1}\phi_k(v_j)\det\begin{pmatrix}\phi_1(v_1)&\cdots&\widehat{\phi_1(v_j)}&\cdots&\phi_1(v_k)\\ \vdots&&\vdots&&\vdots\\ \phi_{k-1}(v_1)&\cdots&\widehat{\phi_{k-1}(v_j)}&\cdots&\phi_{k-1}(v_k)\end{pmatrix}.$$

By the induction hypothesis, we thus have

$$\det(\phi_i(v_j))_{i,j=1}^k=\sum_{j=1}^k(-1)^{k-j}(\phi_1\wedge\dots\wedge\phi_{k-1})(v_1,\dots,\widehat{v_j},\dots,v_k)\phi_k(v_j)$$

which is exactly the definition of $((\phi_1\wedge\dots\wedge\phi_{k-1})\wedge\phi_k)(v_1,\dots,v_k)$ in terms of $(k-1,1)$-shuffles. □

We shall now use Proposition 3.6 to determine bases of spaces of k-forms and in particular the dimensions of the latter.

Corollary 3.7. Let V be an n-dimensional real vector space with basis $(e_1, \dots, e_n)$ and let $(e_1^*, \dots, e_n^*)$ denote the dual basis. Then $(e_{j_1}^* \wedge \cdots \wedge e_{j_k}^*)_{1 \le j_1 < \cdots < j_k \le n}$ is a basis of $\Lambda^k(V^*)$. In particular

$$\dim \Lambda^k(V^*) = \binom{n}{k}.$$

Proof. We show that the vectors of said tuple are linearly independent and that $\Lambda^k(V^*)$ has at most the asserted dimension. To this end, let $1 \le j_1 < \cdots < j_k \le n$ and $1 \le l_1 < \cdots < l_k \le n$. Then by Proposition 3.6,

$$(e_{j_1}^* \wedge \cdots \wedge e_{j_k}^*)(e_{l_1}, \dots, e_{l_k}) = \det \begin{pmatrix} e_{j_1}^*(e_{l_1}) & \cdots & e_{j_1}^*(e_{l_k}) \\ \vdots & & \vdots \\ e_{j_k}^*(e_{l_1}) & \cdots & e_{j_k}^*(e_{l_k}) \end{pmatrix}.$$

Assume that the above expression is non-zero. Then in particular the first row is non-zero which implies $j_1 \in \{l_1, \dots, l_k\}$ by definition of the dual basis. Similarly, $j_i \in \{l_1, \dots, l_k\}$ for all $i \in \{1, \dots, k\}$ and therefore $j_i = l_i$ for all $i \in \{1, \dots, k\}$ by the ordering of the multi-indices. As a consequence, we have $(e_{j_1}^* \wedge \cdots \wedge e_{j_k}^*)(e_{l_1}, \dots, e_{l_k}) = \delta_{j_1 l_1} \cdots \delta_{j_k l_k}$. Suppose now that there is a linear relation

$$\sum_{j_1 < \cdots < j_k} a_{j_1 \cdots j_k} e_{j_1}^* \wedge \cdots \wedge e_{j_k}^* = 0 \qquad (a_{j_1 \cdots j_k} \in \mathbb{R})$$

among the asserted basis vectors. Then evaluation on $(e_{l_1}, \dots, e_{l_k})$ shows $a_{l_1 \cdots l_k} = 0$ by the above.

Now we show that $\Lambda^k(V^*)$ has at most the asserted dimension. For this, it suffices to show that the linear map

$$e : \Lambda^k(V^*) \to \mathbb{R}^{\binom{n}{k}}, \ \omega \mapsto (\omega(e_{j_1}, \dots, e_{j_k}))_{1 \le j_1 < \cdots < j_k \le n}$$

is injective: Suppose $e(\omega) = 0$ and let $v_1, \dots, v_k \in V$. It suffices to show that $\omega(v_1, \dots, v_k) = 0$. Indeed, we compute

$$\begin{aligned} \omega(v_1, \dots, v_k) &= \omega\left(\sum_j v_{1j} e_j, \dots, \sum_j v_{kj} e_j\right) \\ &= \sum_{j_1, \dots, j_k} v_{1 j_1} \cdots v_{k j_k} \omega(e_{j_1} \cdots e_{j_k}) = \sum_{\substack{j_1, \dots, j_k \\ \text{distinct}}} v_{1 j_1} \cdots v_{k j_k} \omega(e_{j_1} \cdots e_{j_k}) \end{aligned}$$

The last sum vanishes since for every k-tuple $j_1, \dots, j_k \in \{1, \dots, n\}$ of distinct numbers there is a permutation $\sigma \in S_k$ such that $j_{\sigma(1)} < \cdots < j_{\sigma(k)}$ and therefore $\omega(e_{j_1}, \dots, e_{j_k}) = \operatorname{sign}(\sigma) \omega(e_{j_{\sigma(1)}}, \dots, e_{j_{\sigma(k)}})$ which vanishes by assumption. □

Recall that binomial coefficients are the entries of Pascal's triangle. In particular, $\Lambda^k(V^*) = 0$ for $k > \dim V$ and $\Lambda^k(V^*) = \Lambda^{\dim V - k}(V^*)$ for $k \in \{0, \dots, \dim V\}$.

$$\begin{matrix} & & & 1 & & & \\ & & 1 & & 2 & & 1 \\ & 1 & & 3 & & 3 & & 1 \\ & & & \vdots & & & \end{matrix}$$

Remark 3.8. Before organizing k-forms on a manifold into a bundle we remark on contravariance and coordinate-free definitions of the determinant of a matrix and the cross-ratio of four lines.

(i) Let V, W be vector spaces and let $T : V \to W$ be a linear map. Further, let $T^* : W^* \to V^*$ denote the dual map of T. Then there is an induced map $\Lambda^k T^* : \Lambda^k(W^*) \to \Lambda^k(V^*)$ defined by

$$\Lambda^k T^*(\alpha)(v_1, \ldots, v_k) := \alpha(Tv_1, \ldots, Tv_k)$$

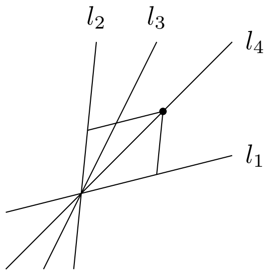
where $\alpha \in \Lambda^k(W^*)$ and $v_1, \ldots, v_k \in V$.

(ii) Let V be an n-dimensional real vector space and let $T \in \mathrm{End}(V)$. Then $\Lambda^n T^* \in \mathrm{End}(\Lambda^n(V^*))$. Since $\Lambda^n(V^*)$ is one-dimensional, so is its endomorphism algebra. Furthermore, the endomorphism algebra of a one-dimensional real vector space E is, in contrast to the space itself, *canonically* isomorphic to $\mathbb{R}$ via $\mathbb{R} \to \mathrm{End}(E)$, $\lambda \mapsto \lambda\,\mathrm{Id}$. Therefore $\Lambda^n T^*$ is canonically associated to a real number. It is an exercise to show that this number is $\det T$.

(iii) To further illustrate the importance of the fact that the endomorphism algebra of a one-dimensional real vector space is *canonically* isomorphic to $\mathbb{R}$ consider the following: In geometry, one defines the cross-ratio of four non-zero complex numbers z_1, z_2, z_3, z_4 by

$$(z_1, z_2; z_3, z_4) := \frac{(z_1 - z_3)(z_2 - z_4)}{(z_2 - z_3)(z_2 - z_4)}$$

Now, identify $\mathbb{C}$ as a two-dimensional real vector space and let l_1, l_2, l_3 and l_4 be the lines through the origin associated to z_1, z_2, z_3 and z_4 respectively. It is a fact that for a two-dimensional vector space V its general linear group $\mathrm{GL}(V)$ acts transitively on the set of lines through the origin. It even acts transitively on the set of triples of lines through the origin. Hence there is no non-trivial linear invariant of such lines. However, it does not act transitively on quadruples of lines through the origin and the cross-ratio provides a linear invariant for such quadruples. Now, consider the map $\varphi_3 : l_1 \to l_2$ given by moving parallel to l_2 towards l_3 and then parallel to l_1 towards l_2. Similarly, define $\varphi_4 : l_1 \to l_2$. Then $\varphi_4^{-1}\varphi_3 \in \mathrm{GL}(l_1) \cong \mathbb{R}^*$ is the cross-ratio.



3.2. **Differential Forms on Manifolds.** Previously we have introduced the space of k-forms $\Lambda^k(V^*)$ of a real finite-dimensional vector space V. We now apply this definition in the context of manifolds to organize the k-forms on the tangent spaces of a manifold M into a bundle: Define

$$\Lambda^k(M) := \bigcup_{x \in M} \{x\} \times \Lambda^k(\mathrm{T}_x M^*).$$

Let $\pi : \Lambda^k(M) \to M$, $(x, \omega) \mapsto x$ denote the projection onto the first factor of $\Lambda^k(M)$. As in the case of the tangent and cotangent bundle we equip $\Lambda^k(M)$ with the structure of a smooth manifold such that π becomes a smooth map which we outline now: Given a coordinate chart (U, φ) on M and $x \in M$, we have an isomorphism of vector spaces $D_{\varphi(x)}\varphi^{-1} : \mathbb{R}^m \to \mathrm{T}_x M$ which induces an isomorphism

$$\Lambda^k((D_{\varphi(x)}\varphi^{-1})^*) : \Lambda^k((\mathrm{T}_x M)^*) \to \Lambda^k((\mathbb{R}^m)^*).$$

Define the topology and smooth structure on Λ^k as before using the bijections

$$\pi^{-1}(U) \to \varphi(U) \times \Lambda^k((\mathbb{R}^m)^*),\ (x, \omega) \mapsto (\varphi(x), \Lambda^k(D_{\varphi(x)}\varphi^{-1})^*(\omega)).$$

Definition 3.9. Let M be a manifold. A *differential k-form on M* is a map $\omega : M \to \Lambda^k(M)$ such that $\pi \circ \omega = \mathrm{id}_M$.

In other words, a differential k-form assigns an alternating k-form ω_x on the tangent space $\mathrm{T}_x M$ to every point $x \in M$, denoted by $\omega_x : \mathrm{T}_x M \times \cdots \times \mathrm{T}_x M \to \mathbb{R}$. Notice that despite the name, there is no regularity assumption in the definition of differential k-form. As a map between smooth manifolds, it can be measurable, continuous, C^k, smooth and so on. We denote by $\Gamma(\Lambda^k(M))$ the *set of differential k-forms on M* and by $\Omega^k(M)$ the *set of smooth k-forms on M*. The latter is going to play a fundamental role in what is to follow. Now, we generalize the structure of sets of alternating forms to the context of manifolds:

(i) (Vector space). Let $\alpha, \beta \in \Gamma(\Lambda^k(M))$ and let $\mu, \nu \in \mathbb{R}$. Define $\mu\alpha + \nu\beta \in \Gamma(\Lambda^k(M))$ by $(\mu\alpha + \nu\beta)_x := \mu\alpha_x + \nu\beta_x$ for all $x \in M$.
(ii) (Module). Let $f : M \to \mathbb{R}$ be a function and $\alpha \in \Gamma(\Lambda^k(M))$. We define $f\alpha \in \Gamma(\Lambda^k(M))$ by $(f\alpha)_x := f(x)\alpha_x$.
(iii) (Algebra). Let $\alpha \in \Gamma(\Lambda^p(M))$ and $\beta \in \Gamma(\Lambda^q(M))$. Then we define $\alpha \wedge \beta \in \Gamma(\Lambda^{p+q}(M))$ by $(\alpha \wedge \beta)_x := \alpha_x \wedge \beta_x$ for all $x \in M$ to the effect that $\Gamma(\Lambda^*(M)) := \bigoplus_{k \geq 0} \Gamma(\Lambda^k(M))$ becomes an associative graded-commutative $\mathbb{R}$-algebra.

A crucial feature of k-forms on a manifold is that they can be pulled back from one manifold to another via a smooth map. This is the basis of the "invariant calculus" on manifolds which is to follow.

Definition 3.10. Let M and N be manifolds and let $f : N \to M$ be a smooth map. Further, let $\omega \in \Gamma(\Lambda^k(M))$. Define $f^*\omega \in \Gamma(\Lambda^k(N))$ by

$$f^*(\omega)_x(v_1, \ldots, v_k) := \omega_{f(x)}(D_x f(v_1), \ldots, D_x f(v_k))$$

for all $x \in N$ and $v_1, \ldots, v_k \in \mathrm{T}_x N$.

The pullback operation behaves well with respect to the algebra structures.

Proposition 3.11. Let M, N and P be manifolds and let $f : N \to M$ and $g : M \to P$ be smooth maps. Then

(i) $f^* : \Gamma(\Lambda^k(M)) \to \Gamma(\Lambda^k(N))$ is $\mathbb{R}$-linear,
(ii) for $\omega_1 \in \Gamma(\Lambda^p(M))$ and $\omega_2 \in \Gamma(\Lambda^q(M))$: $f^*(\omega_1 \wedge \omega_2) = f^*(\omega_1) \wedge f^*(\omega_2)$,
(iii) for $g : M \to \mathbb{R}$ and $\omega \in \Gamma(\Lambda^k(M))$ we have $f^*(g\omega) = (g \circ f)f^*\omega$, and
(iv) $(g \circ f)^*(\omega) = f^*g^*(\omega)$.

The last part of Proposition 3.11 implies that the functor which to a manifold associates its space of k-forms is contravariant.

Proof. (Proposition 3.11). We only prove (ii) and (iv). For (ii), let $x \in N$ and $v_1, \ldots, v_{p+q} \in \mathrm{T}_x N$. Then

$$\begin{aligned}
&f^*(\omega_1 \wedge \omega_2)_x(v_1, \ldots, v_{p+q}) = (\omega_1 \wedge \omega_2)_{f(x)}(D_x f v_1, \ldots, D_x f v_{p+q}) \\
&= (\omega_{1,f(x)} \wedge \omega_{2,f(x)})(D_x f v_1, \ldots, D_x f v_{p+q}) \\
&= \frac{1}{p!q!} \sum_{\sigma \in S_{p+q}} \operatorname{sign}(\sigma) \omega_{1,f(x)}(D_x f v_{\sigma(1)}, \ldots, D_x f v_{\sigma(p)}) \\
&\qquad\qquad\qquad \cdot \omega_{2,f(x)}(D_x f v_{\sigma(p+1)}, \ldots, D_x f v_{\sigma(p+q)}) \\
&= (f^*\omega_1)_x \wedge (f^*\omega_2)_x(v_1, \ldots, v_{p+q}) \\
&= (f^*\omega_1 \wedge f^*\omega_2)_x(v_1, \ldots, v_{p+q})
\end{aligned}$$

which is the assertion of part (ii). Part (iv) is yet another incarnation of the chain rule. Given $\omega \in \Gamma(\Lambda^k(P))$ and $v_1, \ldots, v_k \in \mathrm{T}_x N$ we compute

$$\begin{aligned}
(g \circ f)^*(\omega)_x(v_1, \ldots, v_k) &= \omega_{gf(x)}(D_x(gf)v_1, \ldots, D_x(gf)v_k) \\
&= \omega_{gf(x)}(D_{f(x)} g D_x f(v_1), \ldots, D_{f(x)} g D_x f(v_k)) \\
&= g^*\omega_{f(x)}(D_x f v_1, \ldots, D_x f v_k) \\
&= f^* g^* \omega_x(v_1, \ldots, v_k)
\end{aligned}$$

□

In order to deal with smooth k-forms we are going to need to express them in local coordinates. To this end, recall that on $\mathbb{R}^m$ we have the global coordinate functions $\pi_i : \mathbb{R}^m \to \mathbb{R}$ given by $x \mapsto x_i$ which provide globally defined 1-forms $d\pi_1, \ldots, d\pi_m$. For a coordinate chart (U, φ) on a manifold M we define $dx_i := \varphi^*(d\pi_i)$. Then for every $p \in U$, the 1-forms $(dx_1)_p, \ldots, (dx_m)_p$ form a basis of $(\mathrm{T}_p M)^*$. Let now $\omega \in \Gamma(\Lambda^k(M))$. Then by Corollary 3.7 we have for every $p \in U$ that ω_p can be written as

$$\omega_p = \sum_{i_1 < \cdots < i_k} a_{i_1, \ldots, i_k}(p)(dx_{i_1})_p \wedge \cdots \wedge (dx_{i_k})_p$$

in a unique way. In terms of the algebra structure defined above, this reads

$$\omega|_U = \sum_{i_1 < \cdots < i_k} a_{i_1, \ldots, i_k} \; dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

One often writes I to refer to a multi-index $i_1 < \cdots < i_k$ and defines $a_I := a_{i_1, \ldots, i_k}$ as well as $dx_I := dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ in which case the above is condensed to

$$\omega|_U = \sum_I a_I \; dx_I$$

This expression of a differential k-form in local coordinates allows one to check the smoothness of it in the following way.

Lemma 3.12. Let M be a manifold and let $\omega \in \Gamma(\Lambda^k(M))$. Then ω is smooth if and only if in every coordinate chart (U, φ) all the functions $a_I : U \to \mathbb{R}$ in the local coordinate expression of ω with respect to (U, φ) are smooth.

The proof of Lemma 3.12 is left as an exercise. It requires unravelling the definitions of smoothness and smooth structure on $\Lambda^k(M)$; however, once this is done, the statement is rather tautological. Using Lemma 3.12 we can prove the following.

Proposition 3.13. Let M and N be a manifolds and let $f : N \to M$ be a smooth map. Then the following hold.

(i) The exterior product of smooth k-forms on M is smooth, and
(ii) the pullback via f of a smooth k-form on M is a smooth k-form on N.

Proof. For (i), let $\alpha \in \Omega^p(M)$ and $\beta \in \Omega^q(M)$. Further, let (U, φ) be a coordinate chart on M. Writing α and β in local coordinates with respect to (U, φ) we have $\alpha = \sum_I a_I dx_I$ and $\beta = \sum_J b_J dx_J$ for smooth functions $a_I, b_J : M \to \mathbb{R}$. Therefore

$$\alpha \wedge \beta = \sum_{I,J} a_I b_J \; dx_I \wedge dx_J = \sum_{I \cap J \neq \emptyset} a_I b_J (\pm 1) \; dx_Q$$

where Q is the ordered multi-index corresponding to $I \cup J$. The claim now follows from the fact that the functions $U \to \mathbb{R}$ given by $x \mapsto a_I(x) b_J(x)$ are smooth.

For part (ii), let $\omega \in \Omega^k(M)$. Further, let (V, ψ) and (U, φ) be charts on N and M respectively such that $f(V) \subseteq U$. Then in (U, φ)-local coordinates we have

$$\omega|_U = \sum_I a_I \; dx_I = \sum_{i_1 < \cdots < i_k} a_{i_1, \ldots, i_k} \; dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

Therefore $f^*\omega = \sum_{i_1 < \cdots < i_i} (a_{i_1, \ldots, i_k} \circ f) f^*(dx_{i_1}) \wedge \cdots \wedge f^*(dx_{i_k})$. Since the forms $f^*(dx_{i_1})$ are smooth as pullbacks of 1-forms, part (i) implies the assertion.

We could also have determined a local coordinate expression for the $f^*(dx_i)$. Namely,

$$f^*(dx_i) = f^*(\varphi^*(d\pi_i)) = (\varphi \circ f)^*(d\pi_i) = (\varphi \circ f \circ \psi^{-1} \circ \psi)^*(d\pi_i) = \psi^*(\varphi \circ f \circ \psi^{-1})^*(d\pi_i)$$

Now, let $F : \psi(V) \to \varphi(U)$ denote the map $\varphi \circ f \circ \psi^{-1}$. If further we denote by $p_j : \mathbb{R}^n \to \mathbb{R}$ the coordinate functions on $\mathbb{R}^n$ and by $dp_1, \ldots, dp_n$ the associated 1-forms then with $dy_i := \psi^*(dp_j)$ we have

$$F^*(d\pi_i)_y = \sum_{j=1}^n \frac{\partial F_i}{\partial y_j}(y) \; dp_j$$

and therefore

$$f^*(dx_i)_p = \sum_{j=1}^n \frac{\partial F_i}{\partial y_j}(\psi(p)) \; dy_j$$

which implies the assertion. □

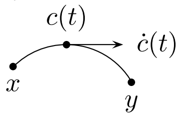
We now compute the pullback of a differential form in a particularly important case which will reappear later on: Let $U, V \subseteq \mathbb{R}^m$ be open and let $f : U \to V$ be smooth. A top form $\omega \in \Omega^m(V)$ can be written as $\omega = a dx_1 \wedge \cdots \wedge dx_m$ where $a : V \to \mathbb{R}$ is a smooth function. To compute $f^*\omega$, note that for dimension reasons we already know that $f^*\omega = g \; dx_1 \wedge \cdots \wedge dx_m$ for some smooth function $g : U \to \mathbb{R}$. To determine $g(p)$ for $p \in U$ we just need to evaluate $(f^*\omega)_p$ on the standard basis $(e_1, \ldots, e_m)$ since $dx_1 \wedge \cdots \wedge dx_m(e_1, \ldots, e_m) = 1$. By definition:

$$\begin{aligned}
(f^*\omega)_p(e_1, \ldots, e_m) &= \omega_{f(p)}(D_p f(e_1), \ldots, D_p f(e_m)) \\
&= a(f(p))(dx_1 \wedge \cdots \wedge dx_m)_{f(p)}(D_p f(e_1), \ldots, D_p f(e_m)) \\
&= a(f(p))((dx_1)_{f(p)} \wedge \cdots \wedge (dx_m)_{f(p)})(D_p f(e_1), \ldots, D_p f(e_m)) \\
&= a(f(p)) \det((dx_i)_{f(p)}(D_p f(e_j)))_{i,j} \\
&= a(f(p)) \det(e_i^*(D_p f(e_j))_{i,j} \\
&= a(f(p)) \det(D_p f).
\end{aligned}$$

Overall, we have $(f^*\omega)_p = a(f(p)) \det(D_p f) \; dx_1 \wedge \cdots \wedge dx_m$. This computation can be generalized to k-forms in which case the coefficients appearing in $f^*\omega$ are determinants of minors of $D_p f$.

3.3. **Partition of Unity.** In this section we construct the most powerful tool for studying smooth manifolds, namely *partition of unity.* It allows one to construct various global objects out of locally defined ones, such as a Riemannian metric and the integral of a suitable differential form over a manifold, denotes $\int_M \omega$. We demonstrate the idea in the case of a Riemannian metric which is a smooth choice of scalar products $(\langle -,- \rangle_x)_{x\in M}$ on the tangent spaces T_xM of a manifold M. Using these we can define the length of a C^1-path $c:[0,1]\to M$ by

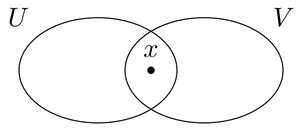
$$l(c) := \int_0^1 \|\dot{c}(t)\|_{c(t)}\ dt$$



This in turn allows us to define the Riemannian distance of points $x,y\in M$ associated to the Riemannian metric by

$$d(x,y) := \inf\{l(c) \mid c:[0,1]\to M\ \ C^1\text{-path with } c(0)=x \text{ and } c(1)=y\}.$$

The problem lies in defining the Riemannian metric in the first place. For instance, if (U,φ) is a chart of M at $x\in M$ one could define $\langle -,-\rangle_x$ as the pullback of the standard scalar product on $\mathbb{R}^m = D_x\varphi \mathrm{T}_xM$. This works well as long as there is only one chart involved. However, if a point $x\in M$ lies in two charts domains U and V we cannot guarantee that the scalar products so defined on T_xM coincide.



However, we may take a convex linear combination. For instance, if $\varphi_U : M\to\mathbb{R}$ and $\varphi_V : M\to\mathbb{R}$ are functions which vanish outside U and V respectively and whose sum at x is one, then $\varphi_U(x)\langle -,-\rangle_{x,U} + \varphi_V\langle -,-\rangle_{x,V}$ is a scalar product on T_xM. This method works as long as there are only finitely many chart domains containing x. In the following we make these difficulties precise and resolve them.

Lemma 3.14. Let M be a topological manifold with a maximal atlas. Further, let $\{V_\alpha \mid \alpha\in A\}$ be a cover of M by open sets. Then there is an at most countable family of charts $\{(U_i,\varphi_i)\mid i\in S\}$ with

(i) $\{U_i \mid i\in S\}$ being a locally finite cover of M refining $\{V_\alpha\mid\alpha\in A\}$, and
(ii) $\varphi_i(U_i)\subseteq C^m_{3\varepsilon}(0)$ as well as $\bigcup_{i\in S} V_i = M$ where $V_i=\varphi_i^{-1}(C^m_\varepsilon(0))$.

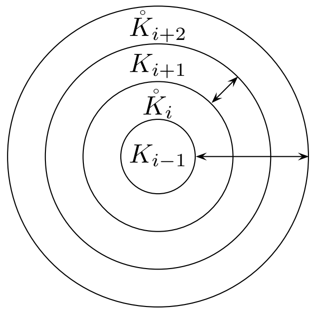
If M is a smooth manifold with atlas the then the charts can be chosen to be diffeomorphisms.

We recall the terminology appearing in Lemma 3.14. A cover $\{U_i\mid i\in S\}$ of a topological space M is *locally finite* if every point in M has a neighbourhood which intersects only finitely many U_i $(i\in S)$. It *refines* another cover $\{V_\alpha\mid\alpha\in A\}$ if for every $i\in S$ there is $\alpha\in A$ such that $U_i\subseteq V_\alpha$.

As an example, consider the real line $\mathbb{R}$ and the cover $\{(-\infty,n)\mid n\in\mathbb{Z}\}$. Then $\{(n-2,n)\mid n\in\mathbb{Z}\}$ is a locally finite subcover refining the former since $(n-2,n)\subset(-\infty,n)$ for all $n\in\mathbb{Z}$ and every point in $\mathbb{R}$ is contained in at most two elements of $\{(n-2,n)\mid n\in\mathbb{Z}\}$. Note, however, that the refining step is necessary: There is no locally finite subcover of $\{(-\infty,n)\mid n\in\mathbb{Z}\}$ itself.

Proof. (Lemma 3.14). We may assume that M is connected and non-compact. Let $\{P_i\mid i\in\mathbb{N}\}$ be a countable basis of open sets such that $\overline{P_i}$ is compact for every $i\in\mathbb{N}$ which exists for the following reason: If $\mathcal{B}=\{B_i\mid i\in\mathbb{N}\}$ is a basis then $\mathcal{B}_c := \{B\in\mathcal{B}\mid \overline{B}\text{ is compact}\}$ works: It is indeed a basis: Since M is locally compact Hausdorff, every B_i is a union of open sets with compact closure and every open set with compact closure is a union of elements of $\mathcal{B}$ which hence lie in $\mathcal{B}_c$. Now, we define an exhaustion of M by compact sets $(K_n)_{n\in\mathbb{N}}$ with nicely behaved interiors and closures which will aid in definining the asserted locally finite refinement of $\{V_\alpha \mid \alpha \in A\}$:

Set $K_1 := \overline{P}_1$. By compactness and the fact that $\{P_i \mid i \in \mathbb{N}\}$ is a basis there is $j \in \mathbb{N}$ such that $P_1 \cup \cdots \cup P_j \supseteq K_1 = \overline{P}_1$. In fact, we must have $j \geq 2$ since there are no non-empty open compact subsets M by the assumption that it be connected and non-compact. Let $r_1 := \min\{\overline{P}_1 \subseteq P_1 \cup \cdots \cup P_j\} \geq 2$ and set $K_2 := \overline{P}_1 \cup \cdots \cup \overline{P}_{r_1}$. Iterate this process to define $K_j := \overline{P}_1 \cup \cdots \cup \overline{P}_{r_{j-1}}$. Then $K_{j-1} \subseteq P_1 \cup \cdots \cup P_{r_{j-1}}$ but $K_{j-1} \not\subseteq P_1 \cup \cdots \cup P_l$ for any $l < r_{j-1}$. Also, observe $\mathring{K}_j \supseteq P_1 \cup \cdots \cup P_{r_{j-1}} \supseteq K_{j-1}$. Therefore, $\mathring{K}_{i+2} \backslash K_{i-1}$ is an open set which contains the compact set $K_{i+1} \backslash \mathring{K}_i$.



Now consider $\mathring{K}_{i+2} \backslash K_{i-1} \cap V_\alpha$: For every $p \in \mathring{K}_{i+2} \backslash K_{i-1} \cap V_\alpha$ let (U_p^i, φ_p^i) be a chart at p contained in $\mathring{K}_{i+2} \backslash K_{i-1} \cap V_\alpha$ and such that $\varphi_p^i(U_p^i) = C_{3\varepsilon}^m(0)$. Then set $V_p^i := (\varphi_p^i)^{-1}(C_\varepsilon^m(0))$. For fixed $i \in \mathbb{N}$, consider the set of all charts (U_p^i, φ_p^i) obtained in this way by varying α over A. Since $\bigcup_{\alpha\in A} V_\alpha = M$ we have in particular that the sets V_p^i so obtained cover $K_{i+1} \backslash \mathring{K}_i$. By compactness we can therefore find a finite set $S_i \subset \mathring{K}_{i+2} \backslash K_{i-1}$ such that $\bigcup_{p\in S_i} V_p^i \supseteq K_{i+1} \backslash \mathring{K}_i$. Now consider the collection $\{(U_p^i, \varphi_p^i) \mid i \in \mathbb{N},\ p \in S_i\}$. First of all, the U_p^i of this collection form a countable refinement of $\{V_\alpha \mid \alpha \in A\}$. Also, $\bigcup_{i\in\mathbb{N}, p\in S_i} V_p^i = M$ by construction. As to local finiteness, let $p \in M$ and $i \in \mathbb{N}$ such that $p \in \mathring{K}_{i-1}$. Observe that for all $j \geq i$ we have $\mathring{K}_{j+2} \backslash K_{j-1} \cap \mathring{K}_{i-1} = \emptyset$ and conclude by noting that for all $j \geq i$ and all $q \in S_j$ we have $U_q^j \subseteq \mathring{K}_{j+2} \backslash K_{j-1}$ whence $U_q^j \cap \mathring{K}_{i-1} = \emptyset$. That is, only finitely many domains of $\{(U_p^i, \varphi_p^i) \mid i \in \mathbb{N},\ p \in S_i\}$ intersect $\mathring{K}_{i-1}$.

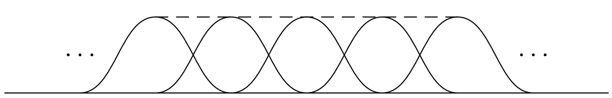
The statement about the smooth case is immediate from the above. □

As a corollary to Lemma 3.14 we have the following.

Theorem 3.15. Let M be a smooth manifold and let $\{V_\alpha \mid \alpha \in A\}$ be an open cover of M. Then there is a countable family $\{f_i \mid i \in F\}$ of functions on M such that

(i) $f_i \geq 0$, $f_i \in C^\infty(M)$ for all $i \in F$ with compact support,
(ii) $\{\operatorname{supp} f_i \mid i \in F\}$ is a locally finite cover of M refining $\{V_\alpha \mid \alpha \in A\}$, and
(iii) $\sum_{i\in F} f_i(x) = 1$ for all $x \in M$.

On the real line, a partition of unity may look as follows.



Proof. (Theorem 3.15). Let $\{(U_i, \varphi)_i \mid i \in I\}$ be a covering as in Lemma 3.14. Now, let $g : \mathbb{R}^m \to \mathbb{R}$ with $0 \leq g(x) \leq 1$ for all $x \in \mathbb{R}^m$ be smooth such that $g \equiv 1$ on

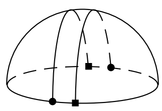
$C^m_\varepsilon(0)$ and $g \equiv 0$ outside $C^m_{2\varepsilon}(0)$. Then set

$$g_i(x) := \begin{cases} g(\varphi_i(x)) & x \in U_i \\ 0 & x \notin U_i \end{cases}$$

Then $\sum_{i\in I} g_i : M \to \mathbb{R}$ is well-defined, smooth and strictly positive everywhere. We may thus set $f_i := g_i/(\sum_{j\in I} g_j)$. □

3.4. **Orientation.** In this section we the introduce the last notion needed to define expressions of the form $\int_M \omega$ where M is a manifold and ω is a top-dimensional form on M, i.e. $\omega \in \Omega^m(M)$ where $m = \dim M$. Namely, M has to be *orientable.* First of all, we recall the notion of orientation on a finite-dimensional real vector space V. We remark that interestingly this notion does depend on the field of scalars. Two bases $(e_1, \ldots, e_n)$ and $(f_1, \ldots, f_n)$ of V are defined to be equivalent if the change of basis matrix $A \in \mathrm{GL}(n, \mathbb{R})$ from $(e_1, \ldots, e_n)$ to $(f_1, \ldots, f_n)$ whose coefficients are given by $f_i = \sum_{j=1}^n a_{ji} e_i$ $(i \in \{1, \ldots, n\})$ has positive determinant. This defines an equivalence relation on the set of bases of V since $\mathrm{GL}^+(n, \mathbb{R}) := \{A \in \mathrm{GL}(n, \mathbb{R}) \mid \det A > 0\}$ is a subgroup of $\mathrm{GL}(n, \mathbb{R})$. An *orientation on* V is a choice of an equivalence class of bases. Hence there are two possible orientations on a given finite-dimensional real vector space. For instance, on $\mathbb{R}^2$, the standard basis (e_1, e_2) is typically said to define the positive orientation and $(e_1, -e_2)$ is said to define the negative orientation. The fact that these bases define different orientations is also reflected by the fact that they cannot be transformed into each other continuously while maintaing the property of forming a basis. This in turn comes from the fact that $\mathrm{GL}(n, \mathbb{R})$ has exactly two connected components.

For a smooth manifold M, an orientation would be a consistent, continuous choice of an orientation on T_pM for every $p \in M$. However, this is not always possible. The lowest dimension in which it is not is two. Indeed, $\mathbb{P}^2\,\mathbb{R}$ is not orientable: Think of $\mathbb{P}^2\,\mathbb{R}$ as the northern hemisphere of a sphere with opposite points on the equator identified. Then the image in $\mathbb{P}^2\,\mathbb{R}$ of the strip depicted in the image below is diffeomorphic to a Möbius strip within $\mathbb{P}^2\,\mathbb{R}$.



Hence $\mathbb{P}^2\,\mathbb{R}$ cannot be orientable. In fact, using the theory to be developed in this section, it is an exercise to show that $\mathbb{P}^n\,\mathbb{R}$ is orientable if and only if n is odd.

Definition 3.16. Let M be a smooth manifold.

(i) An atlas $\mathcal{A}$ on M is *oriented* if for any two overlapping charts $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) in $\mathcal{A}$ the coordinate transformation $\theta_{\beta\alpha} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ has the property that $\det D_x\theta_{\beta\alpha} > 0$ for all $x \in \varphi_\alpha(U_\alpha \cap U_\beta)$.

(ii) The manifold M is *orientable* if it admits an oriented atlas.

Given an oriented atlas $\mathcal{A}$ on a manifold M we obtain for every $p \in M$ a well-defined orientation on T_pM by stipulating that for every $(U, \varphi) \in \mathcal{A}$ with $p \in U$ the map $D_p\varphi : \mathrm{T}_pM \to \mathbb{R}^m$ is orientation-preserving; here we put the standard positive orientation on $\mathbb{R}^m$.

There is a characterization of orientability of manifolds in terms of certain differential forms which better allows for computations. For this, recall that for an m-dimensional real vector space V its space of alternating m-forms $\Lambda^m V^*$ is one-dimensional. However, the bundle of alternating m-forms over a manifold behaves very differently from the trivial bundle with fiber $\mathbb{R}$ whose smooth sections are just the smooth functions on M in the sense that the latter admits constants whereas the first one not necessarily does. Hence the following definition makes sense.

Definition 3.17. Let M be a smooth manifold. A *volume form on* M is a smooth differential m-form ω on M which is nowhere vanishing, i.e. $\omega_x \neq 0$ for all $x \in M$.

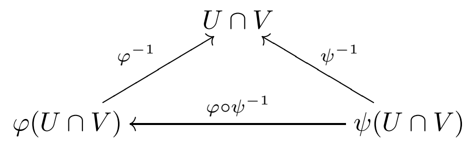
From the remark above it is clear that if ω is a volume form on M then any other volume form on M is of the form $f \cdot \omega$ where $f \in C^\infty(M)$ is nowhere vanishing. The term *volume* form is due to the fact that $\omega_x(e_1, \dots, e_n)$ may be interpreted as the volume of the parallelepiped spanned by the vectors $(e_1, \dots, e_n)$.

Proposition 3.18. Let M be a smooth manifold. Then M is orientable if and only if it admits a volume form.

Proof. Suppose first that M admits a volume form ω. Let $\mathcal{A}$ be an atlas of M. We modify $\mathcal{A}$ into a new atlas $\mathcal{A}'$ which is oriented. First of all, passing to connected components and restricting charts we may assume that all chart domains of $\mathcal{A}$ are connected. Now, given $(U, \varphi) \in \mathcal{A}$ consider $(\varphi^{-1})^*(\omega) = a\ dx_1 \wedge \cdots \wedge dx_m$ where $a : \varphi(U) \to \mathbb{R}$ is a nowhere vanishing smooth function. Then by connectedness of U, the function a has constant sign. If it has positive sign we declare (U, φ) to be part of the new atlas $\mathcal{A}'$. If not, set $(U, s_1 \circ \varphi)$ to be in $\mathcal{A}'$ where $s_1 : \mathbb{R}^m \to \mathbb{R}^m$ is given by $(x_1, \dots, x_m) \mapsto (-x_1, x_2, \dots, x_m)$; in fact any function whose derivative has negative determinant everywhere would do: If we set $\varphi' := s_1 \circ \varphi$ then

$$\begin{aligned}(\varphi'^{-1})^*(\omega) &= (s_1^{-1})^*(\varphi^{-1})^*(\omega) = (s_1^{-1})^*(a\ dx_1 \wedge \cdots \wedge dx_m) = \\ &= -a(-y_1, y_2, \dots, y_m)\ dy_1 \wedge \cdots \wedge dy_m = b\ dy_1 \wedge \cdots \wedge dy_m\end{aligned}$$

on $s_1(\varphi(U))$ where b is of positive sign. We claim that $\mathcal{A}'$ so defined is an oriented atlas on M. It is an atlas simply because its chart domains cover M. To see that it is oriented, let (U, φ) and (V, ψ) in $\mathcal{A}'$ be overlapping charts. Then $(\varphi^{-1})^*(\omega) = a\ dx_1 \wedge \cdots \wedge dx_m$ and $(\psi^{-1})^*(\omega) = b\ dx_1 \wedge \cdots \wedge dx_m$ where a and b are strictly positive functions on $\varphi(U \cap V)$ and $\psi(U \cap V)$ respectively. Furthermore, since the triangle



commutes we have $(\varphi \circ \psi^{-1})^*(\varphi^{-1})^*\omega = (\psi^{-1})^*\omega$ and hence

$$(\varphi \circ \psi^{-1})^*(a\ dx_1 \wedge \cdots \wedge dx_m) = b\ dx_1 \wedge \cdots \wedge dx_m.$$

However, we know from before that $b(p) = a(\varphi \circ \psi^{-1}(p)) \det D_p(\varphi \circ \psi^{-1})$ for all $p \in \psi(U \cap V)$. Hence, since a and b are positive, we deduce that $\det D_p(\varphi \circ \psi^{-1})$ is positive forall $p \in \psi(U \cap V)$. That is, $\mathcal{A}'$ is oriented.

Conversely, suppose that $\mathcal{A}$ is an oriented atlas on M. The idea on how to construct a volume form is to pullback volume forms from chart codomains and to sum the forms so obtained using a partition of unity: Let $(U_i, \varphi_i)_{i \in I}$ be a locally finite refinement of $\mathcal{A}$ and let $\{f_i \mid i \in I\}$ be a partition of unity subordinate to it, i.e. $\operatorname{supp} f_i \subset U_i$, $f_i \in C^\infty(M)$, $0 \leq f_i \leq 1$ and $\sum_{i \in I} f_i(x) = 1$ for all $x \in M$. For every $i \in I$, define $\omega_i \in \Omega^m(M)$ by

$$(\omega_i)_p := \begin{cases} f_i(p)\varphi_i^*(dx_1 \wedge \cdots \wedge dx_m) & x \in U_i \\ 0 & x \notin U_i \end{cases}$$

Now define $\omega := \sum_{i \in I} \omega_i \in \Omega^m(M)$. Then ω is a volume form on M: Given $p \in M$, let $k \in I$ be such that $p \in U_k$. Then ω_p equals the finite sum $\sum_{p \in U_i \cap U_k} (\omega_i)_p$. On $U_i \cap U_k$ we can express ω_i in terms of ω_k as follows: As before, we have

$$\begin{aligned}(\varphi_i)^*(dx_1 \wedge \cdots \wedge dx_m)_p &= (\varphi_k)^*((\varphi_i\varphi_k^{-1})^*(dx_1 \wedge \cdots \wedge dx_m))_p = \\ &= \det D_{\varphi_k(p)}(\varphi_i\varphi_k^{-1})(\varphi_k^*(dx_1 \wedge \cdots \wedge dx_m))_p.\end{aligned}$$

Therefore we have

$$\omega_p = \sum_{p\in U_i\cap U_k} f_i(p)\det D_{\varphi_k(p)}(\varphi_i\circ\varphi_k^{-1})(\varphi_k^*(dx_1\wedge\cdots\wedge dx_m))_p$$

Now observe that since $\varphi_k^*(dx_1\wedge\cdots\wedge dx_m)_p\neq 0$ we only need to show that the coefficients in the above sum are non-zero. Indeed, since $\mathcal{A}$ is oriented we have

$$\sum_{p\in U_i\cap U_k} f_i(p)\det D_{\varphi_k(p)}(\varphi_i\circ\varphi_k^{-1}) \geq \min_{p\in U_i\cap U_k}\det D_{\varphi_k(p)}(\varphi_i\circ\varphi_k^{-1})\underbrace{\sum_{p\in U_i\cap U_k} f_i(p)}_{1}.$$

Hence the assertion. □

Note that the argument of the second part of the proof of Proposition 3.18 works for any manifold up to the point of proving that the coefficients do not vanish, but they will for non-orientable manifolds.

Given a volume form ω on a manifold M, we get a consistent choice of orientation of each T_pM ($p\in M$) by saying that a tuple $(e_1,\dots,e_m)$ of vectors in T_pM is positively oriented if $\omega_p(e_1,\dots,e_m)>0$.

3.5. **Integrating Smooth Compactly Supported Forms And More.** Let M be a smooth manifold with an oriented atlas $\mathcal{A}$ and let $\Omega_c^p(M)$ denote the span in $\Omega^p(M)$ of all smooth p-forms with compact support; recall that for $\omega\in\Omega^p(M)$ we set $\operatorname{supp}(\omega)=\overline{\{x\in M\mid \omega_x\neq 0\}}$. Now, let $(U,\varphi)\in\mathcal{A}$. We define a linear form $I_{(U,\varphi)}:\Omega_c^m(U)\to\mathbb{R}$, where $m=\dim M$, by

$$I_{(U,\varphi)}(\omega):=\int_{\mathbb{R}^m} a(x_1,\dots,x_m)\ d\mu(x_1)\cdots d\mu(x_m)$$

using $a\ dx_1\wedge\cdots\wedge dx_m=(\varphi^{-1})^*(\omega)\in\Omega_c^m(\varphi(U))$, where $a\in C_c^\infty(\mathbb{R}^m)$ has compact support contained in $\varphi(U)$, and the Lebesgue measure, which can be replaced by the Jordan content as long as ω is smooth. The orientability on M is key in proving the following compatibility lemma.

Lemma 3.19. Let M be a manifold with an oriented atlas $\mathcal{A}$. Further, let (U,φ) and (V,ψ) be charts, and $\omega\in\Omega_c^m(U\cap V)$. Then $I_{(U,\varphi)}(\omega)=I_{(V,\psi)}(\omega)$.

Proof. Write $(\varphi^{-1})^*(\omega)=a\ dx_1\wedge\cdots\wedge dx_m$ and $(\psi^{-1})^*(\omega)=b\ dx_1\wedge\cdots\wedge dx_m$. Utilizing that $\varphi\circ\psi^{-1}:\psi(U\cap V)\to\varphi(U\cap V)$ is a diffeomorphism between open subsets of $\mathbb{R}^m$ and the calculation at the end of Section 3.2 we conclude $b(x)=a(\varphi\circ\psi^{-1}(x))\det D_x(\varphi\circ\psi^{-1})$. We therefore have

$$I_{(V,\psi)}(\omega)=\int_{\mathbb{R}^m} b(x)\ d\mu(x)=\int_{\mathbb{R}^m} a(\varphi\circ\psi^{-1}(x))\det D_x(\varphi\circ\psi^{-1})\ d\mu(x).$$

Since $\mathcal{A}$ is oriented we have $\det D_x(\varphi\circ\psi^{-1})>0$ and hence, by the classical change of variables formula with $y:=\varphi\circ\psi^{-1}(x)$, we may continue the above with

$$=\int_{\mathbb{R}^m} a(y)\ d\mu(y)=I_{(U,\varphi)}(w).$$

□

We now extend the definition of integral to the whole manifold: Let $(U_i,\varphi_i,f_i)_{i\in\mathbb{N}}$ be such that the (U_i,φ_i) ($i\in\mathbb{N}$) are contained in the oriented atlas $\mathcal{A}$ and form a locally finite covering, and $(f_i)_{i\in\mathbb{N}}$ is a partition of unity subordinate to $(U_i)_{i\in\mathbb{N}}$. Now given $\omega\in\Omega_c^m(M)$ we set

$$\int_M\omega:=\sum_{i\in\mathbb{N}} I_{(U_i,\varphi_i)}(f_i\omega).$$

Observe that for every $i \in \mathbb{N}$ the form $f_i\omega$ lies in $\Omega_c^m(U_i)$ and that the above sum is finite as only finitely many U_i intersect the compact set $\mathrm{supp}(\omega)$. We proceed by showing that the above sum is independent of the involved choices: If $(V_j, \psi_j, g_j)_{j\in\mathbb{N}}$ is another choice as above we may write $f_i\omega = \sum_{j\in\mathbb{N}} g_j f_i \omega$ and therefore

$$I_{(U_i,\varphi_i)}(f_i\omega) = \sum_j I_{(U_i,\varphi_i)}(g_j f_i\omega) = \sum_{j\in\mathbb{N}} I_{(V_j,\psi_j)}(g_j f_i\omega)$$

where we have used that $\mathrm{supp}(g_j f_i \omega) \subseteq U_i \cap V_j$. Consequently, we have

$$\begin{aligned}\sum_{i\in\mathbb{N}} I_{(U_i,\varphi_i)}(f_i\omega) &= \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} I_{(V_j,\psi_j)}(g_j f_i\omega) \\ &= \sum_{j\in\mathbb{N}}\sum_{i\in\mathbb{N}} I_{(V_j,\psi_j)}(g_j f_i\omega) \\ &= \sum_{j\in\mathbb{N}} I_{(V_j,\psi_j)}\left(\sum_{i\in\mathbb{N}} g_j f_i \omega\right) = \sum_{j\in\mathbb{N}} I_{(V_j,\psi_j)}(g_j\omega).\end{aligned}$$

Overall, $\int_M : \Omega_c^m(M) \to \mathbb{R}$ is a well-defined linear form.

Remark 3.20. Using the Lebesgue integral, the linear form $\int_M$ can be extended to forms ω which are Borel measurable, have compact support and are bounded. Here, Borel measurability refers to ω as a map from M to $\Lambda^k(M)$ and boundedness means the following: Let $\eta \in \Omega^m(M)$ be a volume form. A differential m-form α on M with compact support is *bounded* if there is a constant $c > 0$ such that $|\alpha_x(v_1, \dots, v_m)| \le c|\eta_x(v_1, \dots, v_m)|$ for all $x \in M$ and $v_1, \dots, v_m \in \mathrm{T}_x M$. This definition is independent of the chosen volume form.

This extension of $\int_M$ is important for instance to be able to integrate forms of the type $\chi_D \cdot \omega$ where ω is a smooth form on M and D is some domain.

3.6. **Exterior Derivative.** Having introduced integration of forms we now turn to a differentiation type operation on forms. We have seen that given a smooth manifold M one can define a natural derivative $d : \Omega^0(M) \to \Omega^1(M)$. We are now going to construct natural maps $\Omega^k(M) \to \Omega^{k+1}(M)$ for every value of k which play the role of a derivative. First of all, we define these maps in the case where $M = U$ is an open subset of $\mathbb{R}^m$ and then transplant the result back into manifolds. For $f \in \Omega^0(U) = C^\infty(U)$ we have already defined $df \in \Omega^1(U)$; in our case:

$$df = \sum_{i=1}^m \frac{\partial f}{\partial x_i}\, dx_i.$$

Now, let $k \ge 1$ and $\omega \in \Omega^k(U)$. Then ω can be written in a unique way as $\omega = \sum_I a_I\, dx_I$ where the sum is taken over all ordered multi-indices of length k ranging between 1 and n. For $I = (i_1, \dots, i_k)$ recall that $dx_I := dx_{i_1} \wedge \cdots \wedge dx_{i_k}$.

Definition 3.21. Let $U \subseteq \mathbb{R}^m$ be open and $\omega \in \Omega^k(U)$. Define the *exterior derivative* $d\omega \in \Omega^{k+1}(U)$ of ω by

$$d\omega := \sum_I da_I \wedge dx_I$$

where $\omega = \sum_I a_I\, dx_I$.

Example 3.22. To illustrate that the above formula is far from arbitrary, consider an open subset $U \subseteq \mathbb{R}^2$ and a 1-form $\omega = P\, dx + Q\, dy$ where $P, Q : U \to \mathbb{R}$ are smooth functions. Then

$$\begin{aligned} d\omega &= dP \wedge dx + dQ \wedge dy = \left(\frac{\partial P}{\partial x}\ dx + \frac{\partial P}{\partial y}\ dy\right) \wedge dx + \left(\frac{\partial Q}{\partial x}\ dx + \frac{\partial Q}{\partial y}\ dy\right) \wedge dy \\ &= \frac{\partial P}{\partial x}\ dx \wedge dx + \frac{\partial P}{\partial y}\ dy \wedge dx + \frac{\partial Q}{\partial x}\ dx \wedge dy + \frac{\partial Q}{\partial y} dy \wedge dy \\ &= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\ dx \wedge dy \end{aligned}$$

In this case we recover Green's theorem as $\int_{\partial D} \omega = \int_D d\omega$ which we later identify as an incarnation of Stokes' theorem.

Example 3.23. As a second example, consider an open subset $U \subseteq \mathbb{R}^3$ and the 2-form $\omega := F_1\ dy \wedge dz - F_2\ dx \wedge dz + F_3\ dx \wedge dy$. As above, we compute

$$\begin{aligned} d\omega &= dF_1 \wedge dy \wedge dz - dF_2 \wedge dx \wedge dz + F_3 \wedge dx \wedge dy \\ &= \frac{\partial F_1}{\partial x}\ dx \wedge dy \wedge dz + \frac{\partial F_2}{\partial y}\ dx \wedge dy \wedge dz + \frac{\partial F_3}{\partial z}\ dx \wedge dy \wedge dz \\ &= \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)\ dx \wedge dy \wedge dz. \end{aligned}$$

This resembles the divergence theorem which is yet another incarnation of Stokes' theorem: Indeed, if we consider the vector field $F(x) := (F_1(x), F_2(x), F_3(x))$ on $\mathbb{R}^3$ then $d\omega = \mathrm{div} F\ dx \wedge dy \wedge dz$.

Next, we collect some fundamental properties of the exterior derivative.

Proposition 3.24. Let d be as in Definition 3.21. Then

(i) d is $\mathbb{R}$-linear,
(ii) $d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^{\deg \omega_1} \omega_1 \wedge d\omega_2$ for any two differential forms ω_1 and ω_2 on U,
(iii) $d^2 = 0$, and
(iv) if $f : V \to U$ is a smooth map between open subsets $V \subseteq \mathbb{R}^n$ and $U \subseteq \mathbb{R}^m$ then $df^* = f^* d$.

Note that part (iii) of Proposition 3.24 means that d is not a derivative in the naive sense. Rather it resembles the geometric intuition that the boundary ∂D of a region D does not have a boundary itself, i.e. "$\partial^2 = 0$".

Proof. Part (i) and (ii) are left as exercises but are used. For (iii), suppose first that $f \in \Omega^0(U)$. We show that $d(df)$ vanishes and then consider the general case. This is in fact in incarnation of Schwarz's Theorem:

$$\begin{aligned} d(df) &\overset{\text{(i)}}{=} \sum_{i=1}^m d\left(\frac{\partial f}{\partial x_i}\ dx_i\right) = \sum_{i=1}^m d\left(\frac{\partial f}{\partial x_i}\right) \wedge dx_i = \sum_{i=1}^m \left(\sum_{j=1}^m \frac{\partial^2 f}{\partial x_j \partial x_i}\ dx_j\right) \wedge dx_i \\ &= \sum_{i,j=1}^m \frac{\partial^2 f}{\partial x_j \partial x_i}\ dx_j \wedge dx_i = \sum_{i \neq j} \frac{\partial^2 f}{\partial x_j \partial x_i}\ dx_j \wedge dx_i \\ &= \sum_{i<j} \frac{\partial^2 f}{\partial x_j \partial x_i}\ dx_j \wedge dx_i + \sum_{i>j} \frac{\partial^2 f}{\partial x_j \partial x_i}\ dx_j \wedge dx_i \\ &= \sum_{i<j} \frac{\partial^2 f}{\partial x_j \partial x_i}\ dx_j \wedge dx_i + \sum_{j>i} \frac{\partial^2 f}{\partial x_i \partial x_j}\ dx_i \wedge dx_j \\ &= \sum_{i<j} \left(-\frac{\partial^2 f}{\partial x_j \partial x_i} + \frac{\partial^2 f}{\partial x_i \partial x_j}\right)\ dx_i \wedge dx_j = 0. \end{aligned}$$

Now, if $\omega \in \Omega^k(U)$ for $k \geq 1$, write $\omega = \sum_I a_I \; dx_I$. Then

$$d(d\omega) = d\left(\sum_I da_I \wedge dx_I\right) \overset{\text{(i)}}{=} \sum_I d(da_I \wedge dx_I) \overset{\text{(ii)}}{=} \sum_I \underbrace{d(da_I)}_{=0} \wedge dx_I - da_I \wedge \underbrace{d(dx_I)}_{=0}$$

where $d(da_I) = 0$ by the above and the term $d(dx_I)$ vanishes by recurrence: Recall that $dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$; therefore $d(dx_I) \overset{(i)}{=} d(dx_{i_1} \wedge \cdots \wedge dx_{i_k})$ which equals $d(dx_{i_1}) \wedge (dx_{i_2} \wedge \cdots \wedge dx_{i_k}) - dx_i \wedge d(dx_{i_2} \wedge \cdots \wedge dx_{i_k})$. Here, $d(dx_{i_1})$ vanishes by the discussion for functions and $d(dx_{i_2} \wedge \cdots \wedge dx_{i_k})$ by recurrence. Thus $d(d\omega) = 0$.

For part (iv) we again first consider the case $g \in \Omega^0(U)$. Then

$$f^*(dg) = f^*\left(\sum_{i=1}^m \frac{\partial g}{\partial x_i} \; dx_i\right) = \sum_{i=1}^m f^*\left(\frac{\partial g}{\partial x_i} \; dx_i\right) = \sum_{i=1}^m \left(\frac{\partial g}{\partial x_i} \circ f\right) f^*(dx_i).$$

If $f = (f_1, \ldots, f_m)$ then we have $f^*(dx_i) = \sum_{i=1}^n \frac{\partial f_i}{\partial y_j} \; dy_j$. Therefore:

$$f^*(dg) = \sum_{i=1}^m \left(\frac{\partial g}{\partial x_i} \circ f\right) \sum_{j=1}^n \frac{\partial f_i}{\partial y_j} \; dy_j$$

Now observe that

$$\sum_{i=1}^m \left(\frac{\partial g}{\partial x_i} \circ f\right) \frac{\partial f_i}{\partial y_j}$$

by the chain rule, computing the partial derivative with respect to y_j of the function $y \mapsto g(f_1(y), \ldots, f_m(y))$. Hence

$$f^*(dg) = \sum_{j=1}^m \frac{\partial g \circ f}{\partial y_j} \; dy_j = d(g \circ f) = d(f^* g)$$

which proves the statement for functions $g \in \Omega^0(U)$. Now, if $\omega \in \Omega^k(U)$ and $k \geq 1$ then $\omega = \sum_I a_I \; dx_I$ and we compute

$$\begin{aligned} d(f^*\omega) &= d\left(\sum_I f^*(a_I) f^*(dx_I)\right) = \sum_I d(f^*(a_I) \cdot f^*(dx_I)) \\ &\overset{\text{(ii)}}{=} \sum_I d(f^*(a_I)) \wedge f^*(dx_I) + \sum_I f^*(a_I) d(f^*(dx_I)) \\ &= \sum_I f^*(da_I) \wedge f^*(dx_I) + \sum_I f^*(a_I) d(f^*(dx_I)) \\ &= f^*\left(\sum_I da_I \wedge dx_I\right) + \sum_I f^*(a_I) d(f^*(dx_I)) \\ &= f^*(d\omega) + \sum_I f^*(a_I) \underbrace{d(f^*(dx_I))}_{=0} \end{aligned}$$

where the term $d(f^*(dx_I))$ vanishes by recurrence as before: If $dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ then $f^*(dx_I) = f^* d\pi_{i_1} \wedge \cdots \wedge f^* d\pi_{i_k}$ and applying d yields

$$\begin{aligned} d(f^*(dx_I)) = d(f^* d(\pi_{i_1})) \wedge f^*(d\pi_{i_2}) \wedge \cdots \wedge f^*(d\pi_{i_k}) \\ - f^*(d\pi_{i_1}) \wedge d(f^*(d\pi_{i_2}) \wedge \cdots \wedge f^*(d\pi_{i_k})) \end{aligned}$$

in which $d(f^*(d\pi_{i_1})) = f^*(dd\pi_{i_1}) = 0$ and $d(f^*(d\pi_{i_2}) \wedge \cdots \wedge f^*(d\pi_{i_k}))$ vanishes by recurrence. This proves the assertion. □

We now have an understanding of how the exterior derivate behaves with respect to the various operations for forms defined on open subsets of Euclidean space. Suppose next that M is a manifold and $\omega \in \Omega^k(M)$. We claim that there is a well-defined $(k+1)$-form $d\omega \in \Omega^{k+1}(M)$ such that if $\sum_I a_I \; dx_I$ is the expression of ω in a chart (U, φ) then the expression of $d\omega$ in the same chart is $\sum_I da_I \wedge dx_I$. For this, recall that a_I is a smooth function on U and that $dx_I = \varphi^*(d\pi_I)$ where $d\pi_I = d\pi_{i_1} \wedge \cdots \wedge d\pi_{i_k}$. To see that $d\omega \in \Omega^{k+1}(M)$ is well-defined we rewrite

$$\sum_I da_I \wedge dx_I = \sum_I d(a_I \circ \varphi^{-1} \circ \varphi) \wedge \varphi^*(d\pi_I) = \sum_I d(\varphi^*(a_I \circ \varphi^{-1})) \wedge \varphi^*(d\pi_I)$$
$$= \varphi^* \left(\sum_I d(a_I \circ \varphi^{-1}) \wedge d\pi_I \right) = \varphi^* d \left(\sum_I a_I \circ \varphi^{-1} \; d\pi_I \right) = \varphi^* d((\varphi^{-1})^* \omega)$$

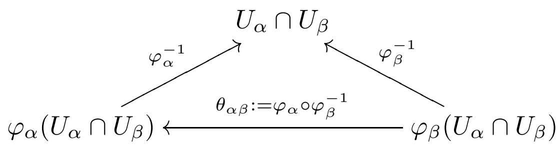
Note that the exterior derivative d in the last expression is indeed applied to a form $(\varphi^{-1})^*\omega$ defined on Euclidean space. In order to show that d is well-defined on manifolds we now show that given any two charts $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) on M we have

$$\varphi_\alpha^*(d(\varphi_\alpha^{-1})^*(\omega)) = \varphi_\beta^*(d(\varphi_\beta^{-1})^*(\omega)).$$

Let $\theta_{\alpha\beta} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$ denote the coordinate transformation from α to β and set $\omega_\alpha := (\varphi_\alpha^{-1})^*\omega$. Then

$$\varphi_\alpha^*(d\omega_\alpha) = (\theta_{\alpha\beta}\varphi_\beta)^*(d\omega_\alpha) = \varphi_\beta^* \theta_{\alpha\beta}^*(d\omega_\alpha) =$$
$$= \varphi_\beta^* d(\theta_{\alpha\beta}^* \omega_\alpha) = \varphi_\beta^*(d(\theta_{\alpha\beta}^*(\varphi_\alpha^{-1})^*\omega)) = \varphi_\beta^*(d(\varphi_\beta^{-1})^*\omega)$$

which is the assertion.



Proposition 3.24 now carries over to the setting of manifolds.

Theorem 3.25. Let M be a manifold and let $d : \Omega^k(M) \to \Omega^{k+1}(M)$ be defined as above. Then

(i) $d : \Omega^0(M) \to \Omega^1(M)$ is the usual differential,
(ii) $d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^{\deg \omega_1} \omega_1 \wedge d\omega_2$ for any two differential forms ω_1 and ω_2 on M,
(iii) $d^2 = 0$, and
(iv) if $F : N \to M$ is smooth map between manifolds then $dF^*\omega = F^*(d\omega)$ for all $\omega \in \Omega^k(M)$.

It is an exercise to convice oneself that these properties of the exterior derivative on manifolds do indeed follow from their counterparts in Euclidean spaces. A more interesting exercise is to show that a collection of maps $\Omega^k(M) \to \Omega^{k+1}(M)$ satisfying the above properties is necessarily given by d.

Before finally turning to Stokes' Theorem we remark on a few invariants of manifolds related to the exterior derivative. Note that given a manifold M we have a sequence

$$0 \xrightarrow{d_{-1}} \Omega^0(M) \xrightarrow{d_0} \Omega^1(M) \xrightarrow{d_1} \cdots \longrightarrow \Omega^{k-1}(M) \xrightarrow{d_{k-1}} \Omega^k(M) \xrightarrow{d_k} \Omega^{k+1}(M) \longrightarrow \cdots$$

Since $d_k \circ d_{k-1} = 0$ we have $\operatorname{im} d_{k-1} \subseteq \ker d_k$. We define the *k-th de Rham cohomology of* M as the quotient vector space $\mathrm{H}^k_{\mathrm{dR}}(M) := \ker d_k / \operatorname{im} d_{k-1}$. For instance, we have $\mathrm{H}^0_{\mathrm{dR}}(M) = \ker d_0 \cong \mathbb{R}^{\pi_0(M)}$ where $\pi_0(M)$ is the number of connected components of M. Thus, in a sense, de Rham cohomology spaces measure higher connectivity problems. Straight from the definition one may also view them as obstructions to solving certain differential equations: Given $\omega \in \Omega^k(M)$, is there a form $\eta \in \Omega^{k-1}(M)$ such that $d\eta = \omega$? A necessary condition is that $d\omega$ be zero. However, this is not sufficient in general.

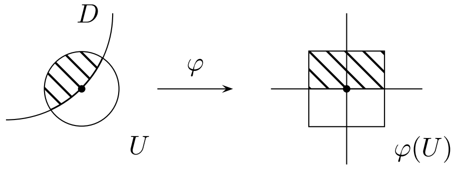
As another fact we state that if M is compact then all its de Rham cohomology spaces are finite-dimensional despite its constituents being uncountable dimensional. For instance, if S_g denotes the surface with g holes then $\mathrm{H}^0_{\mathrm{dR}}(S_g) \cong \mathbb{R}$ since S_g is connected, $\mathrm{H}^2_{\mathrm{dR}}(S_g) \cong \mathbb{R}$ by Poincaré duality since S_g is orientable and $\mathrm{H}^1_{\mathrm{dR}}(S_g) \cong \mathbb{R}^{2g}$. In this case, de Rham cohomology detects the number of holes.

If M is a compact manifold and $S^k(M) = \{\sigma : \Delta^k \to M \text{ smooth}\}$ denotes the set of smooth k-simplices in M then integration against a k-form provides a natural map $S^k(M) \to \mathbb{R}$, $\sigma \mapsto \int_\sigma \omega$. This induces a duality map $\mathrm{H}_k(M, \mathbb{R}) \times \mathrm{H}^k_{\mathrm{dR}}(M) \to \mathbb{R}$ where $\mathrm{H}_k(M, \mathbb{R})$ denotes the singular homology of M over $\mathbb{R}$ computed with respect to smooth k-simplices.

3.7. **Stokes' Theorem.** The general form of Stokes' Theorem requires introducing the concept of smooth manifolds M *with boundary* ∂M and to develop the concepts of smoothness for maps, differential forms etc. in this context. Whereas this does not present any major difficulties, it takes time. A shortcut that we will take is to prove Stokes' Theorem so called *regular domains* in smooth manifold which is less general only in appearance.

Definition 3.26. Let M be a manifold. A *regular domain* in M is a subset $D \subseteq M$ such that

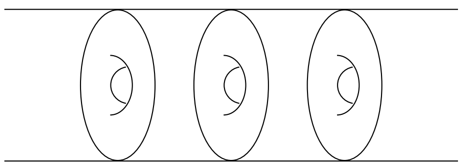
(i) D is closed,
(ii) $\mathring{D}$ is not empty, and
(iii) for every $p \in \partial D = D \setminus \mathring{D}$ there is a chart (U, φ) of M at p such that $\varphi(U) = C^m_\varepsilon(0)$, $\varphi(p) = 0$ and $\varphi(U \cap D) = \{x \in C^m_\varepsilon(0) \mid x_m \geq 0\}$.



Note that Definition 3.26 in particular implies that ∂D is a regular $(m-1)$-submanifold of M.

Example 3.27. The reader is invited to convince himself that the following are examples of regular domains.

(i) $D = \{x \in \mathbb{R}^m \mid \|x\| \leq 1\}$.
(ii) Let N be a smooth manifold and set $M := N \times \mathbb{R}$. Then $N \times [0, 1]$ is a regular domain in M with boundary $\partial D = N \times \{0\} \cup N \times \{1\}$.



Theorem 3.28. Let M be an oriented manifold of dimension at least two and let D be a regular domain in M. Then ∂D is orientable and the orientation of M canonically determines an orientation on ∂D.

Proof. Let $\mathcal{A}$ be the set of all charts as in the definition of a regular domain. Using a volume form on M we can orient $\mathcal{A}$ as in the proof of Proposition 3.18

without changing property (iii) of Definition 3.26: Indeed, either $D_p\varphi : \mathrm{T}_pM \to \mathbb{R}^m$ preserves the orientation in which case we leave it or it reverses the orientation in which case we compose it with the map $\mathbb{R}^m \to \mathbb{R}^m$ which sends $(x_1, x_2, \dots, x_m)$ to $(-x_1, x_2, \dots, x_m)$; observe that this coordinate transformation preserves $\{x \in C_\varepsilon^m(0) \mid x_m \geq 0\}$. Now, let $p \in \partial D$, and (U, φ) and (V, ψ) in $\mathcal{A}$ be charts at p. Then

$$\psi \circ \varphi^{-1}(\{x \in \varphi(U \cap V) \mid x_m \geq 0\}) = \{y \in \psi(U \cap V) \mid y_m \geq 0\}$$

and

$$\psi \circ \varphi^{-1}(\{x \in \varphi(U \cap V) \mid x_m = 0\}) = \{y \in \psi(U \cap V) \mid y_m = 0\}.$$

In particular, $D_0(\psi \circ \varphi^{-1}(x))(\mathbb{R}^{m-1} \times \{0\}) = \mathbb{R}^{m-1} \times \{0\}$. Writing $\psi \circ \varphi^{-1}(x) = (F_1(x), \dots, F_m(x))$ for $x \in \varphi(U \cap V)$ we therefore have

$$D_0(\psi \circ \varphi^{-1}) = \begin{pmatrix} \frac{\partial F_1}{\partial x_1}(0) & \cdots & \frac{\partial F_1}{\partial x_{m-1}}(0) & \frac{\partial F_1}{\partial x_m}(0) \\ \vdots & & \vdots & \vdots \\ \frac{\partial F_{m-1}}{\partial x_1}(0) & \cdots & \frac{\partial F_{m-1}}{\partial x_{m-1}}(0) & \frac{\partial F_{m-1}}{\partial x_m}(0) \\ 0 & \cdots & 0 & \frac{\partial F_m}{\partial x_m}(0) \end{pmatrix} = \begin{pmatrix} M_{m-1} & * \\ 0 & \frac{\partial F_m}{\partial x_m}(0) \end{pmatrix}.$$

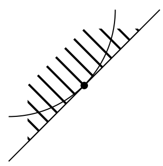
Since $\mathcal{A}$ is oriented we have $\det M_{m-1} \cdot \frac{\partial F_m}{\partial x_m}(0) = \det D_0(\psi \circ \varphi^{-1}) > 0$. From the fact $\psi \circ \varphi^{-1}(\{x \in \varphi(U \cap V) \mid x_m > 0\}) = \{y \in \psi(U \cap V) \mid y_m > 0\}$ we deduce that $\frac{\partial F_m}{\partial x_m}(0) > 0$ which implies $\det M_{m-1} > 0$ by the above. Now, for every $(U, \varphi) \in \mathcal{A}$ define $\widetilde{\varphi} := \varphi|_{U \cap \partial D}$. Since $M_{m-1} = D_0(\widetilde{\varphi} \circ \widetilde{\psi}^{-1})$ we conclude that $\{(U \cap \partial D, \widetilde{\varphi})\}$ is an oriented atlas of ∂D. □

In other words, we take the following from Theorem 3.28.

Scholium. Retain the notation of the proof of Theorem 3.28. Given $p \in \partial D$ and $(U, \varphi) \in \mathcal{A}$ the space

$$\mathrm{T}_pM_{>0} := \{v \in \mathrm{T}_pM \mid (D_p\varphi(v))_m > 0\}$$

is a well-defined half-space, independent of the choice of chart. It contains the "inward" directions.



Now recall that if M is orientable and $\mathcal{A}$ an oriented atlas on M, then an orientation on M is a consistent choice of orientation on T_pM for all $p \in M$. If M is connected there are two possible orientations: the one for which for every choice of $p \in M$ and $(U, \varphi) \in \mathcal{A}$ at p the map $D_p\varphi : \mathrm{T}_pM \to \mathbb{R}^m$ is orientation-preserving and the one for which all these maps are orientation-reversing; here, we equip $\mathbb{R}^m$ with the canonical orientation in which $(e_1, \dots, e_m)$ is positively oriented. Given an orientation on M, the induced orientation on ∂D is defined as follows: Given $p \in \partial D$, a basis $(f_1, \dots, f_{m-1})$ of $\mathrm{T}_p\partial D$ is positively oriented if for all $v \in \mathrm{T}_pM_{>0}$ the basis $(f_1, \dots, f_{m-1}, v)$ of T_pM is positively oriented.

Theorem 3.29. Let M be a smooth oriented manifold and let $D \subseteq M$ be a regular domain. Further, let $\omega \in \Omega^{m-1}(M)$ be a smooth $(m-1)$-form. Assume that either D is compact or that ω has compact support. Then

$$\int_D d\omega = \int_{\widetilde{\partial D}} i^*\omega$$

where $i : \partial D \to M$ denotes the inclusion and $\widetilde{\partial D}$ denotes the boundary of D with the induced orientation if m is even and the opposite of the induced orientation if m is odd.

The proof of Theorem 3.29 reduces to the fundamental theorem of calculus. The reduction however, employs all our knowledge on differential forms.

Proof. Let $(U_i, \varphi_i)_{i\in I}$ be a countable, locally finite covering of M taken from an oriented atlas and such that (U_i, φ_i) is as in Definition 3.26 whenever $U_i \cap \partial D \neq \emptyset$. Furthermore, let $\{f_i \mid i \in I\}$ be a partition of unity subordinate to $(U_i, \varphi_i)_{i\in I}$. We now give a proof in the case where ω has compact support. It constitutes in computing both sides of the stated equality. First of all $\omega = \sum_{i\in I} f_i\omega$ contains only finitely many non-zero terms since $\operatorname{supp}\omega$ is compact. Hence

$$\int_D d\omega = \sum_{j\in I}\int_D d(f_j\omega) \quad \text{and} \quad \int_{\widetilde{\partial D}} i^*\omega = \sum_{j\in I}\int_{\widetilde{\partial D}} i^*(f_j\omega)$$

It therefore suffices to show $\int_D d(f_j\omega) = \int_{\widetilde{\partial D}} i^*(f_j\omega)$ for all $j \in I$ which amounts to showing that

$$\int_{D\cap U_j} d(f_j\omega) = \int_{\widetilde{\partial D\cap U_j}} i^*(f_j\omega).$$

By definition, the left hand side of the above is computed as follows. Consider $\varphi_j : U_j \to C^m_\varepsilon(0)$. Then

$$(\varphi_j^{-1})^*(f_j\omega) = \sum_{l=1}^m g_l \; dx_1 \wedge \cdots \wedge \widehat{dx_l} \wedge \cdots \wedge dx_m$$

for smooth functions $g_j : \mathbb{R}^m \to \mathbb{R}$ with compact support contained in $C^m_\varepsilon(0)$. Using the fact that pulling back commutes with taking the exterior derivative we compute

$$\begin{aligned}
(\varphi_j^{-1})^*(df_j\omega) &= d((\varphi_j^{-1})^*(f_j\omega)) \\
&= \sum_{l=1}^m dg_l \wedge dx_1 \wedge \cdots \wedge dx_l \wedge \cdots \wedge \widehat{dx_l} \wedge \cdots \wedge dx_m \\
&= \sum_{l=1}^m \left(\sum_{s=1}^m \frac{\partial g_l}{\partial x_s}\; dx_s\right) \wedge dx_1 \wedge \cdots \wedge \widehat{dx_l} \wedge \cdots \wedge dx_m \\
&= \sum_{l=1}^m \frac{\partial g_l}{\partial x_l}\; dx_l \wedge dx_1 \wedge \cdots \wedge \widehat{dx_l} \wedge \cdots \wedge dx_m \\
&= \sum_{l=1}^m (-1)^{l-1}\frac{\partial g_l}{\partial x_l}\; dx_1 \wedge \cdots \wedge dx_m.
\end{aligned}$$

We therefore obtain

$$\begin{aligned}
\int_{D\cap U_j} d(f_j\omega) &= \int_{\varphi_j(D\cap U_j)} \sum_{l=1}^m (-1)^{l-1}\frac{\partial g_l}{\partial x_l}\; dx_1\cdots dx_m \\
&= \int_{x_m\geq 0} dx_m \int_{\mathbb{R}^{m-1}} dx_1\cdots dx_{m-1}\left(\sum_{l=1}^m (-1)^{l-1}\frac{\partial g_l}{\partial x_l}\right)
\end{aligned}$$

of which we compute each summand individually: If $l = m$ we have

$$\begin{aligned}
(-1)^{m-1}&\int_{x_m\geq 0} dx_m \int_{\mathbb{R}^{m-1}} dx_1\cdots dx_m \frac{\partial g_m}{\partial x_m} \\
&= (-1)^{m-1}\int_{\mathbb{R}^{m-1}} dx_1\cdots dx_{m-1} \underbrace{\int_{x_m\geq 0}\frac{\partial g_m}{\partial x_m}\; dx_m}_{=-g_m(x_1,\dots,x_{m-1},0)} \\
&= (-1)^m \int_{\mathbb{R}^{m-1}} g_m(x_1,\dots,x_{m-1},0)\; dx_1\cdots dx_{m-1}.
\end{aligned}$$

On the other hand, for $l \neq m$ we get

$$\int_{x_m \geq 0} dx_m \int_{\mathbb{R}^{m-2}} dx_1 \cdots \widehat{dx_l} \cdots dx_{m-1} \underbrace{\int_{\mathbb{R}} dx_l \frac{\partial g_l}{\partial x_l}}_{=0} = 0$$

Overall, we have

$$\int_{D \cap U_j} d(f_j \omega) = (-1)^m \int_{\mathbb{R}^{m-1}} g_m(x_1, \ldots, x_{m-1}, 0)\ dx_1 \cdots dx_{m-1}.$$

We now turn to computing $\int_{\widetilde{\partial D \cap U_j}} i^*(f_j\omega)$: Consider the following diagram

$$\begin{array}{ccc} U_j & \xrightarrow{\varphi_j} & C_\varepsilon^m(0) \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle I} \\ \partial D \cap U_j & \xrightarrow{p_m \circ \varphi_j} & C_\varepsilon^{m-1}(0) \end{array}$$

where $p_m(x_1, \ldots, x_m) := (x_1, \ldots, x_{m-1})$ and $I(x_1, \ldots, x_{m-1}) := (x_1, \ldots, x_{m-1}, 0)$. Then $((p_m \circ \varphi_j)^{-1})^*(i^*(f_j\omega)) = I^*((\varphi_j^{-1})^*(f_j\omega))$. Recall that

$$(\varphi_j^{-1})^*(f_j\omega) = \sum_{l=1}^{m} g_l\ dx_1 \wedge \cdots \wedge \widehat{dx_l} \wedge \cdots \wedge dx_m$$

and note that

$$I^*(dx_1 \wedge \cdots \wedge \widehat{dx_l} \wedge \cdots \wedge dx_m) = I^*(dx_1) \wedge \cdots \wedge \widehat{I^*(dx_l)} \wedge \cdots \wedge I^*(dx_m)$$

where $I^*(dx_m) = 0$ and $I^*(dx_l) = dx_l$ for $l \neq m$. This implies

$$I^*((\varphi_j^{-1})^*(f_j\omega)) = g_m(\ldots, 0)\ dx_1 \wedge \cdots \wedge dx_{m-1}$$

and hence the result; note that if $\partial D \cap U_j = \emptyset$ then clearly $\int_{\widetilde{\partial D \cap U_j}} i^*(f_j\omega) = 0$. Also, $\int_{D \cap U_j} d(f_j\omega)$ vanishes in this case since integration will be over the whole of $\mathbb{R}^m$ without restriction on the x_m-coordinate. □

Corollary 3.30. Let M be an oriented manifold of dimension at least two and let D be a compact regular domain in M. Further, let $\Omega \supseteq D$ be open. Then there is no smooth map $f : \Omega \to \partial D$ which is the identity on ∂D.

Proof. Assume that $f : D \to \partial D$ is as asserted. Pick a volume form on $\omega \in \Omega^{m-1}(\partial D)$ on ∂D. Then $\int_{\partial D} \omega \neq 0$. On the other hand, note that $d^\Omega(f^*\omega) = f^*(d^{\partial D}\omega) = 0$ since by $d^{\partial D}\omega$ is an m-form on ∂D and hence identically zero. Therefore Theorem 3.29 yields a contradiction:

$$0 = \int_D d(f^*\omega) = \int_{\widetilde{\partial D}} i^*(f^*\omega) = \int_{\widetilde{\partial D}} (f \circ i)^*(\omega) \neq 0.$$

□

Corollary 3.31 (Brouwer's Fixed Point Theorem). Let $m \geq 2$ and $\overline{B} := \{x \in \mathbb{R}^m \mid \|x\|^2 \leq 1\}$. Then any continuous map $f : \overline{B} \to \overline{B}$ has a fixed point.

Proof. As a first step, we prove a slightly different statement in a smooth setting: Let $\delta > 0$ and $G : B_{1+\delta} \to \overline{B}$ a smooth map. We claim that G has a fixed point. If $G(x) \neq x$ for all $x \in B(0, 1+\delta)$ then we may consider the ray $G(x) + t(x - G(x))$ $(t \geq 0)$ and define $f(x)$ to be the intersection of this ray with $S(0,1)$. Then $f : B(0, 1+\delta) \to \overline{B}$ is smooth and restricts to the identity on $\partial\overline{B} = S(0,1)$. This, however, contradicts Corollary 3.30.

As a second step, we pass from the smooth to the continuous setting using Weierstrass approximation. Assume that there is a continuos map $F : \overline{B} \to \overline{B}$ with $F(x) \neq x$ for all $x \in \overline{B}$. Then there is $\varepsilon > 0$ such that $0 < 2\varepsilon < \min_{x \in \overline{B}} \|F(x) - x\|$.

By Weierstrass approximation there is a polynomial map $P : \mathbb{R}^m \to \mathbb{R}^m$ such that $\|F(x) - P(x)\| < \varepsilon$ for all $x \in \overline{B}$. Since $\|F(x)\| \leq 1$ we get $\max_{x \in \overline{B}} \|P(x)\| < 1+\varepsilon$. Now define $G : \mathbb{R}^m \to \mathbb{R}^m$ by $G(x) = P(x)/(1+\varepsilon)$. Then $\max_{x\in\overline{B}} \|G(x)\| < 1$. Hence there is $\delta > 0$ such that $G(B(0, 1+\delta)) \subseteq \overline{B}$ and we claim that $G(x) \neq x$ for all $x \in B(0, 1+\delta)$. This is due to the fact that G approximates F well: If $x \in B(0, 1+\delta)\backslash\overline{B}$ then $G(x) \neq x$ since $G(B(0, 1+\delta)) \subseteq \overline{B}$. For $x \in \overline{B}$ we have

$$\|G(x) - F(x)\| \leq \|G(x) - P(x)\| + \|P(x) - F(x)\| < \frac{\varepsilon P(x)}{1+\varepsilon} < 2\varepsilon$$

and therefore $\|G(x) - x\| \geq \|F(x) - x\| - 2\varepsilon > 0$. □

3.8. **De Rham Cohomology in Top Degree.** Let M be a smooth manifold. We have already seen that $\mathrm{H}^0_{\mathrm{dR}}(M)$ consists of locally constant functions and hence can be identified with $\mathbb{R}^{\pi_0}$ where π_0 denotes the number of connected components of M. In this section, we determine $\mathrm{H}^m_{\mathrm{dR}}(M)$ in the case where M is connected, compact and oriented of dimension m. We will see that integration over M, which can be viewed as a linear map from $\Omega^m(M) \to \mathbb{R}$, induces an isomorphism $\mathrm{H}^m_{\mathrm{dR}}(M) \cong \mathbb{R}$. The following two lemmas work towards this statement.

Lemma 3.32. Let M be a manifold and let (U, φ) be a chart of M. Further, suppose that $F \subseteq U$ is such that $\varphi(F) = [0,1]^m$. If $\omega \in \Omega^m(M)$ satisfies $\int_M \omega = 0$ and $\operatorname{supp}\omega \subseteq F$ then there is $\eta \in \Omega^{m-1}(M)$ such that $\operatorname{supp}\eta \subseteq F$ and $\omega = d\eta$.

Proof. Consider the form $(\varphi^{-1})^*\omega \in \Omega^m(\varphi(U))$ with support in $\varphi(F) = [0,1]^m$. Then we can write $(\varphi^{-1})^*\omega = f\ dx_1 \wedge \cdots \wedge dx_m$ for some $f \in C^\infty(\mathbb{R}^m)$ with support in $\varphi(F)$. In this situation, it is an exercise to show that there are $f_i \in C^\infty(\mathbb{R}^m)$ $(i \in \{1, \ldots, m\})$ with support in $\varphi()$ such that $f = \sum_{i=1}^m \frac{\partial f_i}{\partial x_i}$. Now define $\alpha \in \Omega^{m-1}(\mathbb{R}^m)$ by

$$\alpha := \sum_{i=1}^m (-1)^{i-1} f_i\ dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_m$$

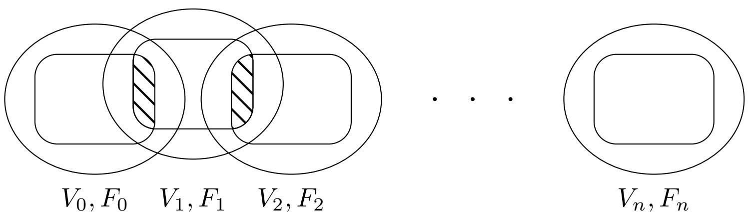
which is supported in $\varphi(F)$. One readily computes that $d\alpha = (\varphi^{-1})^*\omega$ and hence we may put $\eta := \varphi^*\alpha$. □

The second lemma deals with adjusting the support of a general m-form to be contained in a set F as in Lemma 3.32.

Lemma 3.33. Let M be a connected, compact manifold and let (U, φ) be a chart of M such that $F \subset U$ satisfies $\varphi(F) = [0,1]^m$. If $\omega \in \Omega^m(M)$ there is $\eta \in \Omega^{m-1}(M)$ such that $\operatorname{supp}(\omega + d\eta) \subset F$,

Proof. The proof proceedes in two steps.

(i) Claim: Let $(V_i, \varphi_i, F_i)_{i=0}^n$ be such that for every $i \in \{0, \ldots, n\}$ the tuple (V_i, φ_i) is a chart with $F_i \subset V_i$ satisfying $\varphi_i(F_i) = [0,1]^m$. Furthermore, assume that $\mathring{F}_{i-1} \cap \mathring{F}_i \neq \emptyset$ for $1 \leq i \leq n$ and let $\omega_0 \in \Omega^m(M)$ be supported in F_0. Then there is $\eta \in \Omega^{m-1}(M)$ with $\operatorname{supp}(\omega_0 + d\eta) \subseteq F_n$.



Choose m-forms $\omega_1, \dots, \omega_n$ with $\operatorname{supp}\omega_i \subseteq \mathring{F}_{i-1} \cap \mathring{F}_i$ and such that $\int_{\mathring{F}_{j-1}} \omega_j = \int_{\mathring{F}_{j-1}} \omega_{j-1}$ for all $j \in \{1, \dots, n\}$. Then $\operatorname{supp}(\omega_{i+1} - \omega_i) \subseteq F_i$ for $0 \leq i \leq n-1$ and $\int_{V_i} \omega_{i+1} - \omega_i = 0$. Then by Lemma 3.32 there are $\eta_i \in \Omega^{m-1}(M)$ such that $d\eta_i = \omega_{i+1} - \omega_i$ for all $0 \leq i \leq n-1$. Hence

$$\begin{aligned}\omega_n &= (\omega_n - \omega_{n-1}) + (\omega_{n-1} - \omega_{n-2}) + \cdots + (\omega_1 - \omega_0) + \omega_0 \\ &= \sum_{i=0}^{n-1} d\eta_i + \omega_0\end{aligned}$$

Hence the assertion.

(ii) We now finish the proof of the lemma. Let $(U_1, \varphi_1, F_1), \dots, (U_n, \varphi_n, F_n)$ be charts as in step (i) and suppose $(U_1, \varphi_1, F_1) = (U, \varphi, F)$. Furthermore, assume by compactness that $\bigcup_{i=1}^n \mathring{F}_i = M$ and let $(f_i)_{i=1}^n$ be a partition of unity subordinate to $(\mathring{F}_i)_{i=1}^n$. Then we may write $\omega = \sum_{i=1}^n f_i\omega$. Now, for $i \geq 2$ there is, by connectedness and step (i), a form $\eta_i \in \Omega^{m-1}(M)$ such that $\operatorname{supp}(f_i\omega + d\eta_i) \subseteq F_1 = F$. Then $\omega + \sum_{i=1}^n d\eta_i$ has support in F.

□

We are now in a position to prove the following theorem.

Theorem 3.34. Let M be a compact, connected and oriented manifold. Then the sequence

$$\Omega^{m-1}(M) \xrightarrow{d_{m-1}} \Omega^m(M) \xrightarrow{\int_M} \mathbb{R}$$

is exact, i.e. $\ker \int_M = \operatorname{im} d_{m-1}$.

Proof. First, we show that $\operatorname{im} d_{m-1} \subseteq \ker \int_M$. Let $\omega \in \Omega^{m-1}(M)$. Take any chart (U, φ) of M such that the closed unit ball $\overline{B}$ is contained in $\varphi(U)$ and define $D := \varphi^{-1}(\overline{B})$. Then both D and $\overline{M \backslash D}$ are regular domains. Furthermore, they share the same boundary $\partial D = \partial(\overline{M \backslash D})$ but induce opposite orientations on it. As a consequence, Stokes' Theorem implies

$$\int_M d\omega = \int_D d\omega + \int_{\overline{M\backslash D}} d\omega = \int_{\widetilde{\partial D}} \omega + \int_{\widetilde{\partial(\overline{M\backslash D})}} \omega = 0.$$

We now turn to proving that $\ker \int_M \subseteq \operatorname{im} d_{m-1}$. To this end, let $\omega \in \ker \int_M$ and let (U, φ) be a chart such that $F \subseteq U$ satisfies $\varphi(F) = [0,1]^m$. Then by Lemma 3.33 there is $\eta \in \Omega^{m-1}(M)$ with $\operatorname{supp}(\omega + d\eta) \subset F$. Also, $\int_M \omega + d\eta = 0$ by the first step. Hence Lemma 3.32 implies that there is $\alpha \in \Omega^{m-1}(M)$ with $\omega + d\eta = d\alpha$. Thus $\omega = d(\alpha - \eta)$. □

4. De Rham Cohomology

4.1. **Basic Definitions.** Let M be a a smooth manifold of dimension m. Recall the sequence

$$0 \longrightarrow \Omega^0(M) \xrightarrow{d_0} \Omega^1(M) \xrightarrow{d_1} \cdots \xrightarrow{d_{m-2}} \Omega^{m-1}(M) \xrightarrow{d_{m-1}} \Omega^m(M) \xrightarrow{0} \cdots .$$

Since $d^2 = 0$, we have $\operatorname{im} d_{k-1} \subseteq \ker d_k$ for every $k \geq 0$ and we may therefore define

$$\mathrm{H}^k_{\mathrm{dR}}(M) := \ker d_k / \operatorname{im} d_{k-1}$$

Elements of $\operatorname{im} d_{k-1}$ are called *exact forms* and elements of $\ker d_k$ are called *closed forms*. The de Rham cohomology spaces are fundamental invariants of M. So far we know that

(i) $\mathrm{H}^0_{\mathrm{dR}}(M) \cong \mathbb{R}^{\pi_0}$, and
(ii) $\mathrm{H}^m_{\mathrm{dR}}(M) \cong \mathbb{R}$ via integration if M is compact, connected and oriented.

Any differential form $\omega \in \Omega^m(M)$ with $\int_M \omega = 1$ is said to represent the *fundamental class* of M. Next, we describe the functorial properties of de Rham cohomology: Let M, N be manifolds and let $f : M \to N$ be smooth. Combining pullback via f and exterior differentiation we obtain the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^0(N) & \xrightarrow{d_0} & \Omega^1(N) & \xrightarrow{d_1} \cdots \xrightarrow{d_{k-1}} & \Omega^k(N) & \xrightarrow{d_k} & \cdots \\ & & \downarrow f^* & & \downarrow f^* & & \downarrow f^* & & \\ 0 & \longrightarrow & \Omega^0(M) & \xrightarrow{d_0} & \Omega^1(M) & \xrightarrow{d_1} \cdots \xrightarrow{d_{k-1}} & \Omega^k(M) & \xrightarrow{d_k} & \cdots \end{array}$$

Since pullback and exterior differentiation commute, so does the above diagram. As a consequence, we have

$$f^*(\ker d_k^N) \subseteq \ker d_k^M \quad \text{and} \quad f^*(\operatorname{im} d_{k-1}^N) \subseteq \operatorname{im} d_{k-1}^M.$$

Indeed, if for instance $\omega \in \ker d_k^N$ then $d_k^M f^*\omega = f^* d_k^N \omega = 0$ and similarly for the assertion about images. Therefore, f induces for every $k \geq 0$ a map $f^* : \mathrm{H}^k_{\mathrm{dR}}(N) \to \mathrm{H}^k_{\mathrm{dR}}(M)$. If the context is clear, we drop the superscript.

4.2. **The Degree of a Map.** Let M and N be compact, connected and oriented manifolds of dimension m. Further, let $f : M \to N$ be a smooth map. By Theorem 3.34, integration induces isomorphisms

$$\begin{array}{ccc} \mathrm{H}^m_{\mathrm{dR}}(N) & \xrightarrow{f_m^*} & \mathrm{H}^m_{\mathrm{dR}}(M) \\ I \downarrow & & \downarrow I \\ \mathbb{R} & \dashrightarrow & \mathbb{R} \end{array}$$

and hence there is a unique linear map $\mathbb{R} \to \mathbb{R}$ which makes the diagram commute.

Definition 4.1. Let M and N be compact, connected and oriented manifolds of dimension m. Further, let $f : M \to N$ be a smooth map. The *degree of* f is the unique real number $\deg f \in \mathbb{R}$ such that the linear map $\mathbb{R} \to \mathbb{R},\ x \mapsto (\deg f) \cdot x$ makes the above diagram commute.

In the following, we show that the degree of a smooth map when defined is actually an integer. First, we record the following lemma which states that f is close to being a covering map.

Lemma 4.2. Let M and N be compact manifolds of dimension m and let $f : M \to N$ be a smooth map. Let $q \in f(M)$ be a regular value. Then $f^{-1}(q)$ is finite and there is an open connected set $V_q \subseteq N$ containing q and for each $p \in f^{-1}(q)$ an open connected set $U_p \subseteq M$ containing p such that $f|_{U_p} : U_p \to V_q$ is a diffeomorphism.

Proof. Since q is regular, for all $p \in f^{-1}(q)$ the differential $D_p f : \mathrm{T}_p M \to \mathrm{T}_q N$ has full rank and hence, in our case, is an isomorphism. Hence, by the inverse function theorem, there are open neighbourhoods U_p' of $p \in M$ and V_q' of $q \in N$ such that $f|_{U_p'} : U_p' \to V_q$ is a diffeomorphism. In particular, $U_p' \cap f^{-1}(q) = \{p\}$. Therefore $f^{-1}(q)$ is discrete in M and hence finite by compactness of M. Now set V_q to be the connected component of $\bigcap_{p \in f^{-1}(q)} V_q'$ and $U_p := (f|_{U_p'})^{-1}(V_q)$. ☐

Retain the situation of Lemma 4.2 and assume in addition that M and N are oriented. For each $p \in f^{-1}(q)$, the differential $D_p f : \mathrm{T}_p M \to \mathrm{T}_q N$ is an isomorphism. We define the index $I(f, p)$ of f at p by

$$I(f,p) := \begin{cases} 1 & \text{if } D_p f \text{ is orientation-preserving} \\ -1 & \text{if } D_p f \text{ is orientation-reversing} \end{cases}$$

The degree of f can be expressed in terms of indices:

Proposition 4.3. Let M and N be compact, connected, oriented manifolds of dimension m. Further, let $f : M \to N$ be a smooth map. Then either there is no regular value in which case $\deg f = 0$ or $\deg f = \sum_{p\in f^{-1}(q)} I(f,p)$ for any regular value q of f.

As an immediate consequence of Proposition 4.3 we record the following.

Corollary 4.4. Let M and N be compact, connected, oriented manifolds of dimension m. Further, let $f : M \to N$ be a smooth map. Then $\deg f \in \mathbb{Z}$ and $|\deg f| \leq |f^{-1}(q)|$. □

Given a compact, connected, oriented manifold M, it is an interesting question to determine which integers occur as degrees of smooth maps $f : M \to M$. For instance, all integers occur as degrees of smooth maps from the torus to itself but only the degrees -1, 0 and 1 occur for smooth maps from a genus two surface to itself which is deeply related to the fact that the latter admits non-Euclidean geometries whereas the torus does not.

Proof. (Proposition 4.3). If there is no regular value then $f(M)$ has measure zero in N by Sard's Theorem 1.43. In addition, $f(M)$ is compact and hence closed. As a reslt, $f(M) \neq N$ and $N\backslash f(M)$ is non-empty and open. Now, let $\omega \in \Omega^m(N)$ be a representative of the fundamental class of N such that the support of ω is contained in $N\backslash f(M)$. Then $f^*\omega = 0$ and hence $\deg f = \int_M f^*(\omega) = 0$.

Now, assume that a regular value $q \in N$ of f exists. Choose open connected neighbourhoods U_p for each $p \in f^{-1}(q)$ and V_q of q as in Lemma 4.2. Since the sets U_p are connected, $f|_{U_p}$ is either orientation-preserving or orientation-reversing on the whole of U_p. We therefore have

$$\deg f = \int_M f^*(\omega) = \sum_{p\in f^{-1}(q)} \int_{U_p} (f|_{U_p})^*\omega = \sum_{p\in f^{-1}(q)} I(f,p) \int_{V_q} \omega = \sum_{p\in f^{-1}(q)} I(f,p)$$

by the change of variables formula. This proves the assertion. □

4.3. **Poincaré Lemma.** Suppose that M and N are smooth manifolds and that $f : N \to M$ is a smooth map. Then f induces linear maps $f^* : \mathrm{H}^k_{\mathrm{dR}}(M) \to \mathrm{H}^k_{\mathrm{dR}}(N)$ in cohomology in every degree. In this section we work towards the statement that if one smooth map f can be smoothly deformed into another smooth map g then the two induce the same maps in cohomology. For instance, if $M = \mathbb{R}^n$ and $f = \mathrm{id}$ then f can be smoothly deformed via the family $f_t := t \cdot \mathrm{id}$ ($t \in [0,1]$) to the zero map $g = 0$. As a consequence, all de Rham cohomology spaces of $\mathbb{R}^n$ except the degree zero one vanish.

The Poincaré lemma will also play a role in proving the fact that the de Rham cohomology of a manifold can be recovered from combinatorial intersection properties of the elements of certain open covers of the manifold.

We begin with the following easy fact from linear algebra.

Lemma 4.5. Let V be a real finite-dimensional vector space and let $p : \mathbb{R} \times V \to V$ be the projection onto the second factor. Define $\lambda \in (\mathbb{R} \times V)^*$ by $\lambda(t,v) = t$. Then

$$\Lambda^k((\mathbb{R} \times V)^*) = p^*(\Lambda^k(V^*)) \oplus \lambda \wedge p^*(\Lambda^{k-1}(V^*))$$

Proof. Let $e_1, \ldots, e_m$ be a basis of V and $e_1^*, \ldots, e_m^*$ the dual basis. Then the elements $p^*(e_1^*), \ldots, p^*(e_m^*), \lambda$ is a basis of $(\mathbb{R} \times V)^*$. Any k-form on $\mathbb{R} \times V$ is a sum over wedge products of length k of these 1-forms. Each such product either contains λ in which case it is a k-form in the first summand or it does not contain λ in which case it is a k-form in the second summand. □

Now, let M be a smooth manifold. Consider $\mathbb{R}\times M$ and let $\pi : \mathbb{R}\times M \to M$ denote the projection onto M. Define $dt \in \Omega^1(\mathbb{R}\times M)$ by $(dt)_{t,m}(s,u) = s$ for all $s \in \mathbb{R} = \mathrm{T}_t\,\mathbb{R}$ and for all $u \in \mathrm{T}_m M$. We want to decompose a k-form on $\mathbb{R}\times M$ into something which genuinely comes from M and something with the dt-part. The previous lemma implies

$$\Lambda^k((\mathrm{T}_{(t,m)})^*) = (D_{(t,m)}\pi)^*(\Lambda^k(\mathrm{T}_m(M)^*) \oplus dt_{(t,m)} \wedge (D_{(t,m)}\pi)^*(\Lambda^{k-1}(\mathrm{T}_m(M)^*))$$

Therefore, every $\omega \in \Omega^k(\mathbb{R}\times M)$ can be written as $\omega = \omega_1 + dt \wedge \omega_2$ where $\omega_1 \in \Omega^k(\mathbb{R}\times M)$, $\omega_2 \in \Omega^{k-1}(\mathbb{R}\times M)$ and

$$(\omega_1)_{(t,m)} \in (D_{(t,m)}\pi)^*(\Lambda^k(\mathrm{T}_m(M)^*)) \quad \text{and} \quad (\omega_2)_{(t,m)} \in (D_{(t,m)}\pi)^*(\Lambda^{k-1}(\mathrm{T}_m(M)^*)).$$

For every $t \in \mathbb{R}$ let $\Omega_2(t) \in \Omega^{k-1}(M)$ be such that

$$(\omega_2)_{(t,m)} = (D_{(t,m)}\pi)^*(\Omega_2(t)_m).$$

In this way, we get a well-defined map from $\mathbb{R}$ to $\Omega^{k-1}(M)$ given by $t \mapsto \Omega_2(t)$. We define

$$I(\omega) = \int_0^1 \Omega_2(t)\ dt.$$

More precisely, for every $m \in M$ and all $v_1, \ldots, v_{k-1} \in \mathrm{T}_m M$ we have

$$I(\omega)_m(v_1, \ldots, v_{k-1}) = \int_0^1 \Omega_2(t)_m(v_1, \ldots, v_{k-1})\ dt.$$

The Poincaré lemma now reads as follows.

Lemma 4.6. Retain the above notation. Let $i_t : M \to \mathbb{R}\times M,\ (m \mapsto (t,m))$. Then for every $\omega \in \Omega^k(\mathbb{R}\times M)$ we have

$$i_1^*(\omega) - i_0^*(\omega) = d_M(I(\omega)) + I(d_{\mathbb{R}\times M}\omega).$$

Proof. We compute both sides of the asserted equality in local coordinates. Let (U,φ) be a chart on M. Then $(\mathbb{R}\times U, \mathrm{id}\times\varphi)$ is a chart on $\mathbb{R}\times M$. Recall that in such local coordinates we express differential forms on M in terms of dx_I where I is a multi-index and dx_I is a form on U. Lifting them to $\mathbb{R}\times U$ via $\pi : \mathbb{R}\times M \to M$ we can express a k-form $\omega \in \Omega^k(\mathbb{R}\times M)$ on $\mathbb{R}\times U$ as follows:

$$\omega = \underbrace{\sum_{|I|=k} a_I(t,x)\pi^*(dx_I)}_{\omega_1} + dt \wedge \underbrace{\sum_{|J|=k-1} b_J(t,x)\pi^*(dx_J)}_{\omega_2}.$$

As a consequence, $\Omega_2(t) = \sum_{|J|=k-1} b_J(t,x) dx_J$. We now look for the ω_2-part in $d\omega = d\omega_1 - dt \wedge d\omega_2$. To this end, compute

$$\begin{aligned} d\omega_1 &= \sum_{|I|=k} \left(\frac{\partial a_I}{\partial t}\ dt + \sum_{i=1}^m \frac{\partial a_I}{\partial x_i}\ \pi^*(dx_i) \right) \wedge \pi^*(dx_I) \\ &= \sum_{|I|=k} \frac{\partial a_I}{\partial t}\ dt \wedge \pi^*(dx_I) + \text{"stuff not containing } dt\text{"} \end{aligned}$$

and

$$\begin{aligned} d\omega_2 &= \sum_{|J|=k-1} \left(\frac{\partial b_J}{\partial t}\ dt + \sum_{j=1}^m \frac{\partial b_J}{\partial x_j}\ \pi^*(dx_j) \right) \wedge \pi^*(dx_J) \\ &= \text{"stuff containing } dt\text{"} + \sum_{|J|=k-1} \sum_{j=1}^m \frac{\partial b_J}{\partial x_j}\ \pi^*(dx_j \wedge dx_J). \end{aligned}$$

We therefore have

$$d\omega = \text{“terms in } dx\text{”} + dt \wedge \left(\sum_{|I|=k} \frac{\partial a_I}{\partial t}\, \pi^*(dx_I) - \sum_{|J|=k-1} \sum_{j=1}^{n} \frac{\partial b_J}{\partial x_j}\, \pi^*(dx_j \wedge dx_J \right)$$

and

$$I(d\omega) = \int_0^1 \sum_{|I|=k} \frac{\partial a_I}{\partial t}\, dx_I\ dt - \int_0^1 dt \sum_{|J|=k-1} \left(\sum_{j=1}^{m} \frac{\partial b_J}{\partial x_j}\, dx_j \wedge dx_J \right)$$

whence

$$d(I(\omega)) = d\int_0^1 \left(\sum_{|J|=k-1} b_J(t,x)\ dx_J \right) dt = \int_0^1 \left(\sum_{|J|=k-1} \left(\sum_{j=1}^{m} \frac{\partial b_J}{\partial x_j}\, dx_j \right) \wedge dx_J \right) dt.$$

Overall, we conclude

$$I(d\omega) + d(I\omega) = \sum_{|I|=k} a_I(1,x)\ dx_I - \sum_{|I|=k} a_I(0,x)\ dx_I = i_1^*(\omega) - i_0^*(\omega).$$

□

As mentioned in the beginning, one important application of Poincaré's Lemma is that *homotopic* maps induce the same map in cohomology. This is made precise in the following.

Definition 4.7. Let M and N be manifolds and let $f_0, f_1 : N \to M$ be smooth maps. A *homotopy between* f_0 *and* f_1 is a smooth map $h : \mathbb{R} \times N \to M$ such that $h(0,m) = f_0(m)$ and $h(1,m) = f_1(m)$ for all $m \in N$. In this situation, f_0 and f_1 are *homotopic*.

Example 4.8. Let $N = M = \mathbb{R}^n$, $f_1 = \mathrm{id}$ and $f_0 = 0$. Then $h : \mathbb{R} \times N \to M$ given by $h(t,x) = t \cdot x$ is a homotopy between f_0 and f_1.

Proposition 4.9. Let M and N be manifolds and let $f_0, f_1 : N \to M$ be homotopic maps. Then $f_0^*, f_1^* : \mathrm{H}^*_{\mathrm{dR}}(M) \to \mathrm{H}^*_{\mathrm{dR}}(N)$ coincide.

Proof. Let $h : \mathbb{R} \times N \to M$ be a homotopy between f_0 and f_1 and consider the following diagram

$$\begin{array}{ccccc}
\Omega^{k-1}(M) & \xrightarrow{h^*} & \Omega^{k+1}(\mathbb{R} \times N) & \xrightarrow{I} & \Omega^k(N) \\
d \big\uparrow & & \big\uparrow d & & \big\uparrow d \\
\Omega^k(M) & \xrightarrow[h^*]{} & \Omega^k(\mathbb{R} \times N) & \xrightarrow[I]{} & \Omega^{k-1}(N)
\end{array}$$

Let $\alpha \in \ker d_k$. We need to show that $f_1^*(\alpha)$ and $f_0^*(\alpha)$ differ only by an exact form. To this end, we apply Poincaré's Lemma 4.6 to $\omega := h^*(\alpha)$:

$$i_1^*(\omega) - i_0^*(\omega) = dI(\omega) + I(d\omega).$$

This reads

$$f_1^*\alpha - f_0^*\alpha = (h \circ i_1)^*\alpha - (h \circ i_0)^*\alpha = dI(h^*\alpha) + I(dh^*\alpha) = dI(h^*\alpha) + Ih^*d\alpha = dIh^*\alpha$$

which is the assertion. □

Definition 4.10. Let M and N be manifolds. Then M and N are *homotopy equivalent* if there are smooth maps $f : M \to N$ and $g : N \to M$ such that $g \circ f$ is homotopic to id_M and $f \circ g$ is homotopic to id_N.

Corollary 4.11. Let M and N be manifolds. If M and N are homotopy equivalent then $\mathrm{H}^k_{\mathrm{dR}}(M) \cong \mathrm{H}^k_{\mathrm{dR}}(N)$ for all degrees k.

Definition 4.12. Let M be a manifold. If M is connected and homotopy equivalent to a point then M is *contractible*.

Corollary 4.13.

(i) $\mathrm{H}^k_{\mathrm{dR}}(\mathbb{R}^n) = 0$ if $k \geq 1$ and $\mathrm{H}^0_{\mathrm{dR}}(\mathbb{R}^n) = \mathbb{R}$.
(ii) Let M be a manifold. Then $\mathrm{H}^k_{\mathrm{dR}}(\mathbb{R}^n \times M) \cong \mathrm{H}^k_{\mathrm{dR}}(M)$ for all degrees k.

For later use we also recall the following consequence of the proof of Lemma 4.6.

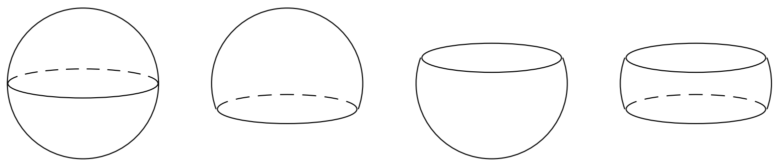
Corollary 4.14. Let M be contractible to $m_0 \in M$. Then there are linear maps $\mathrm{H}_k : \Omega^k(M) \to \Omega^{k-1}(M)$ for all degrees k such that

(i) $\alpha = dH_k(\alpha) + H_{k+1}(d\alpha)$ for all $\alpha \in \Omega^k(M)$ with $k \geq 1$, and
(ii) $\alpha - \alpha(m_0) = H_1(d\alpha)$ for all $\alpha \in \Omega^0(M)$.

Proof. Let $h : \mathbb{R} \times M \to M$ be a homotopy between $f_1 = \mathrm{id}_M$ and the map f_0 with constant value m_0, and go through the proof of Lemma 4.6. □

Retain the notation of Corollary 4.14. If $\alpha \in \Omega^k(M)$ is closed, i.e. $d\alpha = 0$ then $H_k(\alpha)$ is a primitive for α.

4.4. **Mayer-Vietoris Sequence.** Let M be a manifold and write $M = U \cup V$ for open sets $U, V \subseteq M$. In this section we develop Mayer-Vietoris sequences which relate the de Rham cohomologies of M, U, V and $U \cap V$. For instance consider the sphere S^2 and let U and V be small open enlargements of the northern and southern hemisphere respectively. Then both U and V are diffeomorphic to $\mathbb{R}^2$, hence contractible and therefore cohomologically inexistent. The intersection $U \cap V$ on the other hand is diffeomorphic to $\mathbb{R} \times S^1$ whence has the same de Rham cohomology as S^1. The Mayer-Vietoris sequence will enable us to turn this observation into a precise induction argument to compute the de Rham cohmology of spheres.



Definition 4.15. Let A, B and C be vector spaces and let $f : A \to B$ and $g : B \to C$ be linear maps. The sequence

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is *exact at B* if $\operatorname{im} f = \ker g$. An infinite sequence

$$\cdots \longrightarrow V_{n-1} \xrightarrow{f_{n-1}} V_n \xrightarrow{f_n} V_{n+1} \xrightarrow{f_{n+1}} \cdots$$

of vector spaces and linear maps is *exact* if it is exact at every V_n

As an example of exact sequences, we remark that the sequence

$$0 \longrightarrow V \xrightarrow{f} W \quad (V \xrightarrow{f} W \longrightarrow 0)$$

is exact if and only if f is injective (surjective).

Now, let M be a manifold and write $M = U \cup V$ for open sets $U, V \subseteq M$. We have the canonical injections $i_{U,M}$ of U into M, $i_{V,M}$ of V into M, $i_{U\cap V,U}$ of $U \cap V$ into U and $i_{U\cap V,V}$ of $U \cap V$ into V. To avoid an overload of notation we often abbreviate e.g. $i^*_{U,M}(\omega) =: \omega|_U$ for a form on M.

Theorem 4.16. Now, let M be a manifold and write $M = U \cup V$ for open sets $U, V \subseteq M$. For every $k \geq 0$, the sequence

$$0 \longrightarrow \Omega^k(M) \xrightarrow{i^*_{U\oplus V}} \Omega^k(U) \oplus \Omega^k(V) \xrightarrow{i^*_{V\ominus U}} \Omega^k(U \cap V) \longrightarrow 0,$$

where $i^*_{U\oplus V}(\omega) := (i^*_{U,M}(\omega), i^*_{V,M}(\omega))$ for all $\omega \in \Omega^k(M)$, and $i^*_{V\ominus U}(\alpha,\beta) := i^*_{U\cap V,V}(\beta) - i^*_{U\cap V,U}(\alpha)$ for all $\alpha \in \Omega^k(U)$ and $\beta \in \Omega^k(V)$, is exact.

Proof. We need to show exactness at the three slots $\Omega^k(M)$, $\Omega^k(U) \oplus \Omega^k(V)$ and $\Omega^k(U \cap V)$. As to the first one, it is immediate that $i^*_{U\oplus V}$ is injective: If a k-form on M vanishes on both U and V then it vanishes identically as U and V cover M.

Consider now the slot $\Omega^k(U) \oplus \Omega^k(V)$. It is clear that $\operatorname{im} i^*_{U\oplus V} \subseteq \ker i^*_{V\ominus U}$. In order to see the converse inclusion, let $(\alpha,\beta) \in \Omega^k(U) \oplus \Omega^k(V)$ such that $(\alpha,\beta) \in \ker i^*_{V\ominus U}$, i.e. $\alpha|_{U\cap V} = \beta|_{U\cap V}$. Thus we may define $\omega \in \Omega^k(M)$ by setting $\omega|_U = \alpha$ and $\omega|_V = \beta$ to the effect that $i^*_{U\oplus V}(\omega) = (\alpha,\beta)$.

Finally, we show that $i^*_{V\ominus U}$ is surjective: Given $\omega \in \Omega^k(U \cap V)$, we construct $(\omega_U, \omega_V) \in \Omega^k(U) \oplus \Omega^k(V)$ such that $i^*_{V\ominus U}(\omega_U,\omega_V) = \omega$. Let f_U and f_V be a partition of unity subordinate to the cover (U, V) of M and define $\omega_V \in \Omega^k(V)$ by

$$\begin{cases} \omega_V|_{U\cap V} := f_U \cdot \omega \\ \omega_V|_{V\setminus(U\cap V)} = 0 \end{cases}.$$

In order to verify that ω_V is smooth, it suffices to cover its domain V by open sets and show that ω_V is smooth on each of these open sets. In this case, we have $V = (U \cap V) \cup (V \cap (\operatorname{supp} f_U)^c)$ and indeed both $\omega_V|_{U\cap V} = f_U\omega$ and $\omega_V|_{V\cap(\operatorname{supp} f_U)^c} = 0$ are smooth. Similarly, we may define a smooth $\omega_U \in \Omega^k(U)$ by

$$\begin{cases} \omega_U|_{U\cap V} = -f_V \cdot \omega \\ \omega_U|_{U\setminus(U\cap V)} = 0 \end{cases}.$$

We obtain $\omega_V|_{U\cap V} - \omega_U|_{U\cap V} = \omega$. □

For ease of notation we abbreviate

$$0 \longrightarrow \Omega^k(M) \xrightarrow{i_*} \Omega^k(U) \oplus \Omega^k(V) \xrightarrow{p_*} \Omega^k(U \cap V) \longrightarrow 0$$

and denote the induced maps in cohomology by

$$\mathrm{H}^k_{\mathrm{dR}}(M) \xrightarrow{i^*} \mathrm{H}^k_{\mathrm{dR}}(U) \oplus \mathrm{H}^k_{\mathrm{dR}}(V) \xrightarrow{p^*} \mathrm{H}^k_{\mathrm{dR}}(U \cap V)$$

The reason we do not include the zero maps in the sequence in cohomology is that i^* need not be injective and p^* need not be surjective: For instance, $\mathrm{H}^2_{\mathrm{dR}}(S^2) = \mathbb{R}$ but $\mathrm{H}^2_{\mathrm{dR}}(U)$ and $\mathrm{H}^2_{\mathrm{dR}}(V) = 0$. Also, in degree one, we have $\mathrm{H}^1_{\mathrm{dR}}(U)$ and $\mathrm{H}^1_{\mathrm{dR}}(V) = 0$ but $\mathrm{H}^1_{\mathrm{dR}}(U \cap V) \cong \mathrm{H}^1_{\mathrm{dR}}(S^1) \cong \mathbb{R}$.

Nonetheless, the sequence in cohomology is exact at the middle slot.

Lemma 4.17. Retain the above notation. Then $\operatorname{im} i^* = \ker p^*$.

Proof. We know that $p^* \circ i^* = 0$. Hence $\operatorname{im} i^* \subseteq \ker p^*$. For the opposite inclusion, let $(\alpha,\beta) \in \Omega^k(U) \oplus \Omega^k(V)$ with $d\alpha = 0 = d\beta$. The assumption that the class $([\alpha],[\beta]) \in \mathrm{H}^k_{\mathrm{dR}}(U) \oplus \mathrm{H}^k_{\mathrm{dR}}(V)$ is in the kernel of p^* means that $\beta|_{U\cap V} - \alpha|_{U\cap V} = d\gamma$ for some $\gamma \in \Omega^{k-1}(U \cap V)$. Let $(\alpha',\beta') \in \Omega^{k-1}(U) \oplus \Omega^{k-1}(V)$ with $\beta'|_{U\cap V} - \alpha'|_{U\cap V} = \gamma$. Hence $d\beta'|_{U\cap V} - d\alpha'|_{U\cap V} = d\gamma = \beta|_{U\cap V} - \alpha|_{U\cap V}$. Now let $\omega \in \Omega^k(M)$ be such that $\omega|_U = \alpha - d\alpha'$ and $\omega|_V = \beta - d\beta'$ and finish by observing that

$$\begin{cases} d\omega|_U = d(\alpha - d\alpha') = d\alpha = 0 \\ d\omega|_V = d(\beta - d\beta') = d\beta = 0 \end{cases}$$

whence $d\omega = 0$. Hence ω defines a cohomology class in degree k which by construction satisfies $i^*([\omega]) = ([\alpha],[\beta])$. □

We now concern ourselves with the question to which extent the above sequence in cohomology fails to be exact in the first and the third slot. For instance, let $\omega \in \Omega^k(U\cap V)$ represent an element of $\mathrm{H}^k_{\mathrm{dR}}(U\cap V)$, i.e. $d\omega = 0$. Since the sequence

$$0 \longrightarrow \Omega^k(M) \xrightarrow{i_*} \Omega^k(U)\oplus\Omega^k(V) \xrightarrow{p_*} \Omega^k(U\cap V) \longrightarrow 0$$

is exact, there is $(\alpha,\beta) \in \Omega^k(U)\oplus\Omega^k(V)$ such that $p_*(\alpha,\beta) = \omega$. However, there is no reason for α and β to be exact which would imply that p^* is surjective. The obstruction to this is captured by the following diagram

$$\begin{array}{ccccccc}
 & & \Omega^k(U)\oplus\Omega^k(V) & \xrightarrow{p_*} & \Omega^k(U\cap V) & \longrightarrow & 0\\
 & & \big\downarrow{\scriptstyle (d,d)} & & \big\downarrow{\scriptstyle d} & & \\
0 \longrightarrow \Omega^{k+1}(M) & \xrightarrow{i_*} & \Omega^{k+1}(U)\oplus\Omega^{k+1}(V) & \xrightarrow{p_*} & \Omega^{k+1}(U\cap V) & &
\end{array}$$

We have $p_*\circ(d,d)(\alpha,\beta) = d\circ p_*(\alpha,\beta) = d\omega = 0$. Hence there is $\widetilde{\omega} \in \Omega^{k+1}(M)$ such that $i_*\widetilde{\omega} = (d\alpha, d\beta)$. Note that the choice of the pre-image (α,β) of ω is not unique. However, once a choice of (α,β) has been made, $\widetilde{\omega}$ is determined uniquely by injectivity of i_* in the second row.

Furthermore, observe that $d\widetilde{\omega} = 0$ since $d\widetilde{\omega}|_U = d(\widetilde{\omega}|_U) = d(d\alpha) = 0$ and $\widetilde{\omega}|_V = d(\widetilde{\omega}|_V) = d(d\beta) = 0$. This is important as our aim is the construction of a well-defined map $\mathrm{H}^k_{\mathrm{dR}}(U\cap V) \to \mathrm{H}^{k+1}_{\mathrm{dR}}(M)$ that captures the amount to which the map $p^* : \mathrm{H}_{\mathrm{dR}}(U)\oplus\mathrm{H}_{\mathrm{dR}}(V) \to \mathrm{H}_{\mathrm{dR}}(U\cap V)$ fails to be exact. To this end, we directly analyze what happens in case we pick $\omega' \in \Omega^k(U\cap V)$ representing the same cohomology class as ω, i.e. assuming that there is $\eta \in \Omega^{k-1}(U\cap V)$ with $\omega' = \omega + d\eta$. Thus consider the extended diagram

$$\begin{array}{ccccccc}
 & & \Omega^{k-1}(U)\oplus\Omega^{k-1}(V) & \xrightarrow{p_*} & \Omega^{k-1}(U\cap V) & \longrightarrow & 0\\
 & & \big\downarrow{\scriptstyle (d,d)} & & \big\downarrow{\scriptstyle d} & & \\
 & & \Omega^k(U)\oplus\Omega^k(V) & \xrightarrow{p_*} & \Omega^k(U\cap V) & \longrightarrow & 0\\
 & & \big\downarrow{\scriptstyle (d,d)} & & \big\downarrow{\scriptstyle d} & & \\
0 \longrightarrow \Omega^{k+1}(M) & \xrightarrow{i_*} & \Omega^{k+1}(U)\oplus\Omega^{k+1}(V) & \xrightarrow{p_*} & \Omega^{k+1}(U\cap V) & &
\end{array}$$

Since its first row is exact there is $(a,b) \in \Omega^{k-1}(U)\oplus\Omega^{k-1}(V)$ with $p_*(a,b) = \eta$. For the same reason, we may choose pre-images (α,β) of ω and (α',β') of ω'. Then $(\alpha,\beta) + (da,db)$ is a pre-image of ω' by commutativity. Therefore, $(\alpha' - \alpha - da, \beta' - \beta - db)$ is in the kernel of p_*. Hence there is a unique element $c \in \Omega^k(M)$ such that $i_*c = (\alpha' - \alpha - da, \beta' - \beta - db)$. This implies that dc and $\widetilde{\omega}' - \widetilde{\omega}$ map to $(d\alpha' - d\alpha, d\beta' - d\beta)$. This shows that we obtain a well-defined *connecting homomorphism* $\delta : \mathrm{H}^k_{\mathrm{dR}}(U\cap V) \to \mathrm{H}^{k+1}_{\mathrm{dR}}(M)$. Overall, the short exact sequences from above now fit into a long *Mayer-Vietoris* sequence

$$
\begin{array}{l}
0 \longrightarrow \mathrm{H}^0_{\mathrm{dR}}(M) \xrightarrow{i^*} \mathrm{H}^0_{\mathrm{dR}}(U) \oplus \mathrm{H}^0_{\mathrm{dR}}(V) \xrightarrow{p^*} \mathrm{H}^0_{\mathrm{dR}}(U \cap V) \xrightarrow{\delta} \\
\longrightarrow \mathrm{H}^1_{\mathrm{dR}}(M) \xrightarrow{i^*} \mathrm{H}^1_{\mathrm{dR}}(U) \oplus \mathrm{H}^1_{\mathrm{dR}}(V) \xrightarrow{p^*} \mathrm{H}^1_{\mathrm{dR}}(U \cap V) \xrightarrow{\delta} \\
\quad - - - \\
\longrightarrow \mathrm{H}^k_{\mathrm{dR}}(M) \xrightarrow{i^*} \mathrm{H}^k_{\mathrm{dR}}(U) \oplus \mathrm{H}^k_{\mathrm{dR}}(V) \xrightarrow{p^*} \mathrm{H}^k_{\mathrm{dR}}(U \cap V) \xrightarrow{\delta}
\end{array},
$$

which is actually exact.

Theorem 4.18. Let M be a manifold and assume that U, V are open subsets of M such that $M = U \cup V$. Then the Mayer-Vietoris sequence above is exact.

The proof of Theorem 4.18 is not difficult but requires to “deconfuse” oneself and hence constitutes a good exercise which is left to the reader.

Corollary 4.19. Let $m \geq 1$ and $k \in \mathbb{N}_0$. Then

$$
\mathrm{H}^k_{\mathrm{dR}}(S^m) = \begin{cases} \mathbb{R} & k \in \{0, m\} \\ 0 & \text{otherwise} \end{cases}.
$$

Proof. Clearly, we have $\mathrm{H}_{\mathrm{dR}}(S^m) = 0$ for $k \geq m+1$. For the remainder, we may assume $m \geq 2$ and write $S^m = U \cup V$ where $U = S^m \backslash \{S\}$ and $V = S^m \backslash \{N\}$. Then $U \cap V \cong \mathbb{R} \times S^{m-1}$. Now, consider the first part of the long exact sequence for $M = S^m$. It reads

$$
\begin{array}{ccccccccccc}
0 \to & \mathrm{H}^0_{\mathrm{dR}}(S^m) & \to & \mathrm{H}^0_{\mathrm{dR}}(U) \oplus \mathrm{H}^0_{\mathrm{dR}}(V) & \to & \mathrm{H}^0_{\mathrm{dR}}(U \cap V) & \to & \mathrm{H}^1_{\mathrm{dR}}(S^m) & \to & \mathrm{H}^1_{\mathrm{dR}}(U) \oplus \mathrm{H}^1_{\mathrm{dR}}(V) \\
& \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \| & & \| \\
0 \longrightarrow & \mathbb{R} & \xrightarrow{i^*} & \mathbb{R} \oplus \mathbb{R} & \xrightarrow{p^*} & \mathbb{R} & \xrightarrow{\delta} & \mathrm{H}^1_{\mathrm{dR}}(S^m) & \longrightarrow & 0
\end{array}
$$

Indeed, U and V are both contractible, hence the assertion about their cohomology in degree zero and one. Furthermore, $U \cap V \cong \mathbb{R} \times S^{m-1}$ has the same cohomology as S^{m-1} which is connected since $m \geq 2$. Now, the image of i^* equals the kernel of p^* which is hence one-dimensional. Therefore, the image of p^* is of dimension $2 - 1 = 1$, i.e. p^* is surjective. Therefore δ is the zero map. However, δ is also surjective by exactness. This implies $\mathrm{H}^1_{\mathrm{dR}}(S^m) = 0$.

Now, for $m \geq 2$ and $j \geq 1$ we have

$$
\begin{array}{ccccccc}
\mathrm{H}^j_{\mathrm{dR}}(U) \oplus \mathrm{H}^j_{\mathrm{dR}}(V) & \longrightarrow & \mathrm{H}^j_{\mathrm{dR}}(U \cap V) & \xrightarrow{\delta} & \mathrm{H}^{j+1}_{\mathrm{dR}}(S^m) & \longrightarrow & \mathrm{H}^{j+1}_{\mathrm{dR}}(U) \oplus \mathrm{H}^{j+1}_{\mathrm{dR}}(V) \\
\| & & \downarrow\cong & & \| & & \| \\
0 & \longrightarrow & \mathrm{H}^j_{\mathrm{dR}}(S^{m-1}) & \xrightarrow{\delta} & \mathrm{H}^{j+1}_{\mathrm{dR}}(S^m) & \longrightarrow & 0
\end{array}
$$

which implies $\mathrm{H}^j_{\mathrm{dR}}(S^{m-1}) \cong \mathrm{H}^{j+1}_{\mathrm{dR}}(S^m)$. We now have all the information we need to fill in the following tableau by recurrence.

	$\mathrm{H}^0_{\mathrm{dR}}$	$\mathrm{H}^1_{\mathrm{dR}}$	$\mathrm{H}^2_{\mathrm{dR}}$	$\mathrm{H}^3_{\mathrm{dR}}$	$\mathrm{H}^4_{\mathrm{dR}}$	$\cdots$
S^1	$\mathbb{R}$	$\mathbb{R}$	0	0	0	$\cdots$
S^2	$\mathbb{R}$	0	$\mathbb{R}$	0	0	$\cdots$
S^3	$\mathbb{R}$	0	0	$\mathbb{R}$	0	$\cdots$
S^4	$\mathbb{R}$	0	0	0	$\mathbb{R}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$

Hence the assertion. □

4.5. **De Rham cohomology of T^n.** In this section we compute the de Rham cohomology of the $T^n := (S^1)^n$. This can be done using the long exact sequence introduced in the previous section, see [BT03] for an account of this. We shall present a different approach pertaining to the geometry and topology of homogeneous spaces of Lie groups. These methods could have been applied to compute the de Rham cohomology of spheres as well.

Theorem 4.20. There is a natural isomorphism of $\mathrm{H}^k_{\mathrm{dR}}(T^n)$ and $\Lambda^k((\mathbb{R}^n)^*)$.

As a consequence of Theorem 4.20 we obtain $\dim \mathrm{H}^k_{\mathrm{dR}}(T^n) = \binom{n}{k}$. For the remainder of this section shall think of S^1 as the unit circle $S^1 = \{z \in \mathbb{C} \mid |z| = 1\} = \{e^{2\pi i t} \mid t \in \mathbb{R}\}$. Then S^1 and $T^n = (S^1)^n$ are abelian groups. In addition, for every $\xi \in T^n$ the map $L_\xi : T^n \to T^n,\ \eta \mapsto \xi\eta$ is a diffeomorphism. In this context, a k-form $\omega \in \Omega^k(T^n)$ is *invariant* if $L_\xi^* \omega = \omega$ for all $\xi \in T^n$. Let $\Omega^k_{\mathrm{inv}}(T^n)$ denote the subspace of $\Omega^k(T^n)$ of invariant forms and let $\mathbb{1} = (1, \ldots, 1) \in T^n$ denote the identity element.

Lemma 4.21. The map $\Omega^k_{\mathrm{inv}}(T^n) \to \Lambda^k((\mathrm{T}_{\mathbb{1}} T^n)^*)$, $\omega \mapsto \omega_{\mathbb{1}}$ is an isomorphism.

Proof. We construct an inverse to the given map. Let $\varphi \in \Lambda^k((\mathrm{T}_1 T^n)^*)$ and $\xi \in T^n$. We define $\omega \in \Omega^k_{\mathrm{inv}}(T^n)$ by $\omega_\xi(w_1, \ldots, w_k) := \varphi(D_\xi L_{\xi^{-1}}(w_1), \ldots, D_\xi L_{\xi^{-1}}(w_k))$ for all $w_1, \ldots, w_k \in \mathrm{T}_\xi T^n$. One verifies that the k-form ω so defined is invariant and satisfies $\omega_{\mathbb{1}} = \varphi$. □

As a first step towards Theorem 4.20 we show that invariant forms are closed. Then can be checked using local coordinates or with the help of the following trick due to É. Cartan: Consider the exponential map $E : \mathbb{R}^n \to T^n$ given by $x \mapsto (e^{2\pi i x_1}, \ldots, e^{2\pi i x_n})$. This map is a group homomorphism and a local diffeomorphism. In fact, it induces an isomorphism $\mathbb{R}^n / \mathbb{Z}^n \to T^n$. Furthermore, let $s : T^n \to T^n, \xi \mapsto \xi^{-1}$ denote the inversion which is a diffeomorphism of T^n fixing $\mathbb{1}$. We have $D_{\mathbb{1}} s = -\mathrm{Id}$: Indeed, note that $s(E(x)) = E(x)^{-1} = E(-x)$ for all $x \in \mathbb{R}^n$ and hence $(D_0 E)^{-1} \circ D_{\mathbb{1}} s \circ D_0 E = -\mathrm{Id}$ which implies $D_{\mathbb{1}} s = -\mathrm{Id}$.

Lemma 4.22. Let $\omega \in \Omega^k_{\mathrm{inv}}(T^n)$. Then $d\omega = 0$.

Proof. First, observe that s^* preserves $\Omega^k_{\mathrm{inv}}(T^n)$: For all $\xi \in T^n$ we have $s \circ L_\xi = L_{\xi^{-1}} \circ s$ and therefore

$$L_\xi^*(s^*\omega) = (sL_\xi)^*\omega = (L_{\xi^{-1}} s)^*\omega = s^* L_{\xi^{-1}}^* \omega = s^*\omega.$$

for all $\xi \in T^n$, i.e. $s^*\omega$ is invariant.

Next, we compute $s^*\omega$. By invariance, it suffices to determine $s^*\omega$ at $\mathbb{1} \in T^n$. Let $v_1, \ldots, v_k \in \mathrm{T}_{\mathbb{1}} T^n$. Then

$$(s^*\omega)_{\mathbb{1}}(v_1, \ldots, v_k) = \omega_{\mathbb{1}}(D_{\mathbb{1}} s v_1, \ldots, D_{\mathbb{1}} s v_k) = (-1)^k \omega_{\mathbb{1}}(v_1, \ldots, v_k)$$

Hence the invariant forms $s^*\omega$ and $(-1)^k\omega$ conincide everywhere. On the other hand, $d\omega \in \Omega^{k+1}_{\mathrm{inv}}(T^n)$ since differential and pullback commute, therefore

$$(-1)^{k+1} d\omega = s^*(d\omega) = d(s^*\omega) = d((-1)^k \omega = (-1)^k d\omega$$

which implies $d\omega = 0$. □

We aim to show that the combinded map

$$\Omega^k_{\mathrm{inv}}(T^n) \to \ker d_k \to \ker d_k / \operatorname{im} d_{k-1} = \mathrm{H}^k_{\mathrm{dR}}(T^n)$$

is an isomorphism. To this end, we introduce an averaging operator in k-forms which ranges in invariant k-forms: Let $\mathcal{L}$ denote the Lebesgue measure on $\mathbb{R}^n$ with

$\mathcal{L}([0,1]^n) = 1$. Given $\omega \in \Omega^k(T^n)$, we define $P_k\omega \in \Omega^k(T^n)$ by "$\int_{[0,1]^n} L^*_{E(x)}\omega \, d\,\mathcal{L}(x)$", that is, for $\zeta \in T^n$ and $v_1, \ldots, v_k \in \mathrm{T}_\zeta T^n$ we set

$$(P_k\omega)_\zeta(v_1, \ldots, v_k) := \int_{[0,1]^n} (L^*_{E(x)}\omega)_\zeta(v_1, \ldots, v_k) \; d\,\mathcal{L}(x).$$

Lemma 4.23. The operator P_k on $\Omega^k(T^n)$ is a projection ranging in $\Omega^k_{\mathrm{inv}}(T^n)$ and commutes with the differential.

The proof of Lemma 4.23 is left as an exercise. It readily implies that the map

$$\Omega^k_{\mathrm{inv}}(T^n) \to \ker d_k \to \ker d_k / \operatorname{im} d_{k-1} = \mathrm{H}^k_{\mathrm{dR}}(T^n)$$

is injective: Assume that $\omega \in \Omega^k_{\mathrm{inv}}(T^n)$ maps to the trivial class in $\mathrm{H}^k_{\mathrm{dR}}(T^n)$, i.e. there is $\eta \in \Omega^{k-1}(T^n)$ such that $d\eta = \omega$. Then

$$\omega = P_k\omega = P_k(d\eta) = dP_{k-1}\eta = 0$$

since $P_{k-1}\eta$ is invariant and hence closed.

In order to see surjectivity, note that $P_k\omega$ is defined as a "sum" of pullbacks via diffeomorphisms that are all homotopic to the identity, hence $P_k\omega$ should not change the cohomology class of ω. To make this precise, let $x \in \mathbb{R}^n$ and consider the homotopy $H_x : \mathbb{R} \times T^n \to T^n$ given by $(t, \xi) \mapsto E(tx)\xi$. Then $L_{E(x)} = H_x \circ i_1$ and $\mathrm{Id} = H_x \circ i_0$ where $i_t : T^n \to \mathbb{R} \times T^n$ maps ξ to (t, ξ). Furthermore, let $I : \Omega^k(\mathbb{R} \times T^n) \to \Omega^{k-1}(T^n)$ be the operator that occurs in the Poincaré Lemma. Then for every $\alpha \in \Omega^k(\mathbb{R} \times T^n)$ we have $i_1^*\alpha - i_0^*\alpha = dI(\alpha) + I(d\alpha)$. Applying this to $\alpha = (H_x)^*\omega$ with $\omega \in \Omega^k(T^n)$ we get

$$i_1^*(H_1^*)\omega - i_0^*(H_x)^*\omega = dIH_x^*\omega + IdH_x^*\omega$$

and hence $L^*_{E(x)}\omega - \omega = dIH_x^*\omega + IH_x^*\omega$. Integrating over $[0,1]^n$ we finally get

$$P_k\omega - \omega = dK\omega - Kd\omega$$

where $K(\omega) := \int_{[0,1]^n} (IH_x^*)(\omega) \; d(x)$.

Now, let $\omega \in \Omega^k(T^n)$ be such that $d\omega = 0$, i.e. representing a class of $\mathrm{H}^k_{\mathrm{dR}}(T^n)$. Then

$$P_k\omega - \omega = dK\omega + Kd\omega = dK\omega,$$

that is, $P_k\omega \in \Omega^k_{\mathrm{inv}}(T^n)$ and ω represent the same cohomology class. This completes the proof of Theorem 4.20.

We remark that the methods employed to prove Theorem 4.20 can be vastly generalized to the setting of smooth actions of compact connected Lie groups on manifolds: A Lie group G is a smooth manifold G endowed with a group structure such that the multiplication map $G \times G \to G$ and the inversion map $i : G \to G$ are smooth. Now, pick any non-zero alternating form λ on T_eG where $n = \dim G$ and $e \in G$ denotes the identity element. Then $\omega \in \Omega^n(G)$ defined by $\omega_g(v_1, \ldots, v_n) := \lambda(D_gL_{g^{-1}}v_1, \ldots, D_gL_{g^{-1}}v_n)$ for all $v_1, \ldots, v_n \in \mathrm{T}_gG$ is a nowhere vanishing top-form, i.e. a volume form. In particular, G is orientable and admits integration: If $E \subseteq G$ is any relatively compact Borel set, we define $\mathcal{L}(E) := \int_G \chi_E\omega$. Then $\mathcal{L}$ is an invariant regular Borel measure on G, called *Haar measure*. This measure facilitates averaging arguments as in the computation of $\mathrm{H}^*_{\mathrm{dR}}(T^n)$: For instance, if K is a compact connected Lie group and $K \times M \to M$ is a smooth action of K on a manifold M then the set of closed K-invariant k-forms on M surjects onto $\mathrm{H}^k_{\mathrm{dR}}(M)$, i.e. the de Rham cohomology of M can be computed using invariant forms only. For instance, this could have been used to compute $\mathrm{H}^*_{\mathrm{dR}}(S^m)$ using the action of $\mathrm{SO}(m+1)$ on S^m. As another exercise, the reader is invited to show that for any smooth manifold M one has

$$\mathrm{H}^k_{\mathrm{dR}}(S^1 \times M) \cong \mathrm{H}^k_{\mathrm{dR}}(M) \oplus \mathrm{H}^{k-1}_{\mathrm{dR}}(M)$$

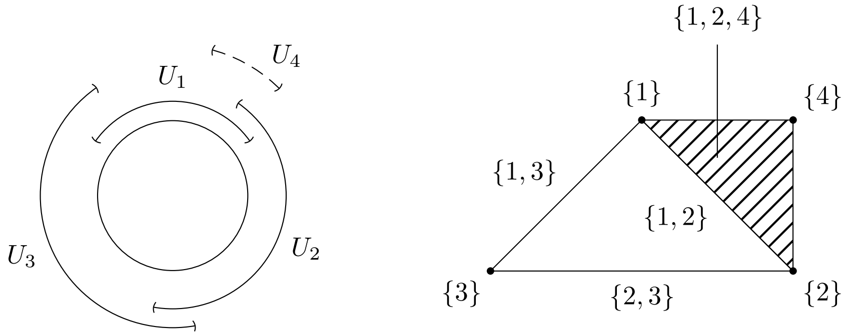
using the action of S^1 on $S^1 \times M$ which acts from the left on S^1 and fixes M and the details of the Poincaré Lemma.

5. De Rham's Theorem

Historically, de Rham's theorem is the first instance of a comparison of different cohomology theories. He showed that for a manifold, de Rham cohomology and singular cohomology, which is defined in the more general context of topological spaces, coincide. The approach to proving this theorem is via Čech cohomology which is combinatorial in nature and can be shown to be equal to singular and de Rham cohomology for any manifold. A classical account of this is [Wei52]. A nice consequence of this is the finite-dimensionality of the de Rham cohomology of a compact manifold in every degree which follows easily from Čech cohomology.

5.1. **Čech Cohomology.** Let X be a set and let $\mathcal{U} = \{U_i \mid i \in I\} \subseteq \mathcal{P}(X)$ be a collection of subsets of X; think of X being a manifold and the U_i being open and covering M. The *nerve* of $\mathcal{U}$ is the set $\mathcal{N}(\mathcal{U})$ of subsets $J \subseteq I$ such that $U_J := \bigcap_{j \in J} U_j \neq \emptyset$. For $q \geq 0$, a *q-simplex* is an ordered $(q+1)$-tuple of indices $\sigma = (i_0, \dots, i_q)$ such that $|\sigma| := \{i_0, \dots, i_q\} \in \mathcal{N}(\mathcal{U})$. The *j-th face* of a q-simplex σ $(q \geq 1)$ is the $q-1$-simplex $\sigma^j = (i_0, \dots, \hat{i}_j, \dots, i_q)$.

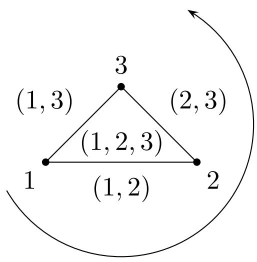
The nerve $\mathcal{N}(\mathcal{U})$ is an instance of what is called a simplicial complex, a combinatorial object for which there is a topological realization. For instance, consider the covering $\{U_1, U_2, U_3\}$ of S^1 indicated below.



If the set U_4 is added to the covering, then a 2-simplex is introduced in the simplicial complex. However, note that this simplex does not change the homotopy type. Now, let $S^q(\mathcal{U})$ denote the set of all q-simplices of $\mathcal{N}(\mathcal{U})$ and let $C^q(\mathcal{U}, \mathbb{R})$ denote the vector space of all functions $f : S^q(\mathcal{U}) \to \mathbb{R}$, termed *q-cochains*. We define the *coboundary operator*

$$\delta_q : C^q(\mathcal{U}, \mathbb{R}) \to C^{q+1}(\mathcal{U}, \mathbb{R}), \ (\delta_q f)(\sigma) = \sum_{j=0}^{q+1} (-1)^j f(\sigma^j)$$

For instance, consider the following simplicial complex.



Then $(\delta_1 f)((1,2,3)) = f((2,3)) - f((1,3)) + f((1,2))$. Thus, in a sense, δ_1 yields a combinatorial boundary of the 2-simplex $(1,2,3)$ with built-in orientation.

Lemma 5.1. Retain the above notation. The map δ_q is linear and $\delta_{q+1} \circ \delta_q = 0$.

As a consequence of the above lemma, we obtain a complex

$$C^0(\mathcal{U},\mathbb{R}) \xrightarrow{\delta_0} C^1(\mathcal{U},\mathbb{R}) \xrightarrow{\delta_1} C^2(\mathcal{U},\mathbb{R}) \xrightarrow{\delta_2} \cdots$$

whose cohomology is the *Čech cohomology*: $\check{\mathrm{H}}^q(\mathcal{U},\mathbb{R}) := \ker \delta_q / \operatorname{im} \delta_{q-1}$. Note that if I is finite then all $C^q(\mathcal{U},\mathbb{R})$ and hence all $\check{\mathrm{H}}^q(\mathcal{U},\mathbb{R})$ are finite-dimensional.

5.2. **Statements.** This section collects the main statements about Čech cohomology. They are proven in the next one. First of all we apply the previous section in the context of manifolds.

Definition 5.2. Let M be a smooth manifold and let $\mathcal{U} = (U_i)_{i\in I}$ be a covering of M by subsets of M. Then $\mathcal{U}$ is *admissible* if

(i) U_i is open for every $i \in I$,
(ii) $\mathcal{U}$ is locally finite, and
(iii) for all $J \in \mathcal{N}(\mathcal{U}, M)$, the intersection $U_J := \bigcap_{i\in J} U_i$ is contractible.

De Rham's Theorem now reads as follows.

Theorem 5.3. Let M be a smooth manifold and let $\mathcal{U}$ be an admissible covering of M. Then for all $k \geq 0$:

$$\mathrm{H}^k_{\mathrm{dR}}(M) \cong \check{\mathrm{H}}^k(\mathcal{U},\mathbb{R}).$$

Concerning the notation $\check{\mathrm{H}}^k(\mathcal{U},\mathbb{R})$ we remark that instead of taking the real numbers as the target of functions on simplices we could have taken any abelian group, e.g. $\mathbb{Z}$. In some cases, this yields more refined invariants. In this case however, it makes sense to compare the $\mathbb{R}$-vector spaces $\check{\mathrm{H}}^k(\mathcal{U},\mathbb{R})$ and $\mathrm{H}^k_{\mathrm{dR}}(M)$.

Note that the statement of Theorem 5.3 is empty without having proven the existence of admissible coverings. This is a theorem on its own.

Theorem 5.4. Let M be a smooth manifold. Then M admits an admissible covering.

One strategy to prove the existence of admissible coverings is to apply Whitney's Embedding Theorem 2.15 first and then argue within Euclidean space. Another one is based on Riemannian geometry and the existence of convex neighbourhoods. Anyway, as a Corollary to Theorem 5.3 we immediately record the following.

Corollary 5.5. Let M be a compact manifold. Then $\mathrm{H}^k_{\mathrm{dR}}(M)$ is finite-dimensional for all $k \geq 0$.

Proof. Let $\mathcal{U}$ be an admissible covering of M and extract a finite subcover. Observe that since $\mathcal{U}$ is finite, so is $S^q(\mathcal{U})$ for every $q \geq 0$ and hence $C^q(\mathcal{U},\mathbb{R})$ is finite-dimensional whence $\check{\mathrm{H}}^k(\mathcal{U},\mathbb{R})$ is finite-dimensional for all $k \geq 0$. □

We remark that in the case of compact manifolds, the above implies that de Rham cohomology can be computed by a machine if an admissible cover and its intersection pattern are provided. This is in sharp contrast to e.g. the fundamental group in which case not even triviality can be decided computationally.

For the next statement recall that we computed

$$\dim \mathrm{H}^k_{\mathrm{dR}}(T^n) = \binom{n}{k} = \binom{n}{n-k} = \dim \mathrm{H}^{n-k}_{\mathrm{dR}}(T^n).$$

This can be generalized to the following extent.

Theorem 5.6 (Poincaré Duality). Let M be a compact oriented manifold. Then the pairing

$$\Omega^k(M) \times \Omega^{n-k}(M) \xrightarrow{\wedge} \Omega^n(M) \xrightarrow{I} \mathbb{R},\ (\alpha, \beta) \mapsto \int_M \alpha \wedge \beta.$$

is non-degenerate and hence $\mathrm{H}^k_{\mathrm{dR}}(M) \cong (\mathrm{H}^{n-k}_{\mathrm{dR}}(M))^*$.

Often, $\dim \mathrm{H}^k_{\mathrm{dR}}(M)$ is referred to as the *k-th Betti number of* M, denoted $b^k(M)$.

5.3. **Proofs.** We now turn to proving de Rham's theorem. Throughout, M denotes a manifold and $\mathcal{U} = (U_i)_{i\in I}$ an admissible covering of M. Recall that $S^p(\mathcal{U})$ denotes the set of all p-simplices, i.e. $(p+1)$-tuples $\sigma = (i_0, \ldots, i_p) \in I^{p+1}$ such that $\bigcap_{\nu=0}^p U_{i_\nu} \neq \emptyset$. In this case, $|\sigma| = \{i_0, \ldots, i_p\}$ denotes the set of vertices.

Definition 5.7. A *Čech form of bidegree* (k, p) $(k, p \geq 0)$ is an $S^p(\mathcal{U})$-tuple $\Omega = (\omega_\sigma)_{\sigma \in S^p(\mathcal{U})}$ with $\omega_\sigma \in \Omega^k(U_{|\sigma|})$. We denote by $\Omega^{(k,p)}$ the real vector space of Čech forms of bidegree (k, p).

For example, a Čech form of bidegree $(k, 0)$ is $\Omega = (\omega_i)_{i\in I}$ with $\omega_i \in \Omega^k(U_i)$. Also, a Čech form of bidegree $(0, p)$ is a tuple $(f_\sigma)_{\sigma\in S^p(\mathcal{U})}$ where $f_\sigma : U_{|\sigma|} \to \mathbb{R}$ are smooth functions. We define

$$d : \Omega^{(k,p)} \to \Omega^{(k+1,p)},\ (d\Omega)_\sigma = d(\omega_\sigma)$$

Consider in particular the map $d : \Omega^{(0,p)} \to \Omega^{(1,p)}$. In the following we identify the kernel of this map: Suppose $\Omega = (f_\sigma)_{\sigma\in S^p(\mathcal{U})} \in \ker d$, i.e. $df_\sigma = 0$ for all $\sigma \in S^p(\mathcal{U})$. Since $U_{|\sigma|}$ is connected, this implies that f_σ takes a constant value depending only on σ. Therefore, an element $\Omega \in \ker(d)$ can be viewed as an element of $C^p(\mathcal{U}, \mathbb{R})$. In the following we introduce differentials $\delta : \Omega^{(k,p)} \to \Omega^{(k,p+1)}$ which together with the differentials d fit into the following complex

$$\begin{array}{ccccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \iddots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
\Omega^2(M) & \longrightarrow & \Omega^{(2,0)} & \longrightarrow & \Omega^{(2,1)} & \longrightarrow & \Omega^{(2,2)} & \longrightarrow & \Omega^{(2,3)} & \longrightarrow & \ldots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
\Omega^1(M) & \longrightarrow & \Omega^{(1,0)} & \longrightarrow & \Omega^{(1,1)} & \longrightarrow & \Omega^{(1,2)} & \longrightarrow & \Omega^{(1,3)} & \longrightarrow & \ldots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
\Omega^0(M) & \longrightarrow & \Omega^{(0,0)} & \longrightarrow & \Omega^{(0,1)} & \longrightarrow & \Omega^{(0,2)} & \longrightarrow & \Omega^{(0,3)} & \longrightarrow & \ldots \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
 & & C^0(\mathcal{U}, \mathbb{R}) & \longrightarrow & C^1(\mathcal{U}, \mathbb{R}) & \longrightarrow & C^2(\mathcal{U}, \mathbb{R}) & \longrightarrow & C^3(\mathcal{U}, \mathbb{R}) & \longrightarrow & \cdots
\end{array}$$

The horizontal differentials δ are defined as follows. First, $\delta : \Omega^k(M) \to \Omega^{(k,0)}$ is defined by $\omega \mapsto (\omega|_{U_i})_{i\in I}$. For $p > 0$ we mimic the boundary operator in the Čech complex:

$$\delta : \Omega^{(k,p)} \to \Omega^{(k,p+1)},\ \Omega = (\omega_\sigma)_{\sigma\in S^p(U)} \mapsto \delta\Omega = (\omega'_\eta)_{\eta\in S^{p+1}(\mathcal{U})} \in \Omega^{(k,p+1)}$$

where $\eta = (i_0, \ldots, i_{p+1}) \in S^{p+1}(M)$ and $\omega'_\eta = \sum_{\nu=0}^{p+1} (-1)^\nu \omega_{(i_0,\ldots,\hat{i}_\nu,\ldots,i_{p+1})}|_{U_{|\eta|}}$. Recall that $U_{|\eta|} = \bigcap_{l=1}^{p+1} U_{i_l}$ which is indeed contained in the support of $\omega_{(i_0,\ldots,\hat{i}_\nu,\ldots,i_{p+1})}$ for all ν. As an example, consider $\Omega^{(k,0)} = \{\Omega = (\omega_i)_{i\in I} \mid \omega_i \in \Omega^k(U_i)\}$ where $I = S^0(\mathcal{U})$. Then $\delta\Omega = (\omega'_{i_0,i_1})_{(i_0,i_1)\in S^1(\mathcal{U})}$ where

$$\omega'_{(i_0,i_1)} = \omega_{i_1}|_{U_{i_0}\cap U_{i_1}} - \omega_{i_0}|_{U_{i_0}\cap U_{i_1}}.$$

This resembles the restriction map in the Mayer-Vietoris sequence.

Lemma 5.8. Retain the above notation. All squares in the above diagram commute.

Proof. Let $\Omega = (\omega_\sigma)_{\sigma \in S^p(\mathcal{U})} \in \Omega^{(k,p)}$ and consider $\delta\Omega := (\omega'_\eta)_{\eta \in S^{p+1}(\mathcal{U})}$. By definition,

$$\omega'_\eta = \sum_{\nu=0}^{p+1} (-1)^\nu \omega_{\eta^\nu}|_{U_{|\eta|}}$$

where η^ν is the ν-th face of η, i.e. $\eta^\nu = (i_0, \ldots, \hat{i}_\nu, \ldots, i_{p+1})$. Now applying d to $\delta\Omega$ yields $d\delta\Omega = (d\omega'_\eta)_{\eta \in S^{p+1}(\mathcal{U})}$ where

$$d\omega'_\eta = \sum_{\nu=0}^{p+1} (-1)^\nu d(\omega_{\eta^\nu}|_{U_{|\eta|}}) = \sum_{\nu=0}^{p+1} (-1)^\nu (d\omega_{\eta^\nu})|_{U_{|\eta|}}$$

On the other hand, $\delta d\Omega = \delta(d\omega_\sigma)_{\sigma \in S^p(\mathcal{U})} = (\omega''_\eta)_{\eta \in S^{p+1}(\mathcal{U})}$ where

$$\omega''_\eta = \sum_{\nu=0}^{p+1} (-1)^\nu (d\omega_{\eta^\nu})|_{U_{|\eta|}}.$$

Hence the assertion. □

5.3.1. *Homotopies.* All vertical and horizontal complexes in the above diagram are in fact exact. This is the content of the two lemmas in this section. Since for every $\sigma \in S^p(\mathcal{U})$ the set $U_{|\sigma|}$ is contractible, the Poincaré Lemma 4.6 yields a linear map $I_{|\sigma|} : \Omega^m(U_{|\sigma|}) \to \Omega^{m-1}(U_{|\sigma|})$ $(m \geq 1)$ such that $\omega = dI_{|\sigma|}\omega + I_{|\sigma|}d\omega$ for all $\omega \in \Omega^m(U_{|\sigma|})$. Now define

$$I : \Omega^{m,p} \to \Omega^{m-1,p}, \ (\omega_\sigma)_{\sigma \in S^p(\mathcal{U})} \to (I_{|\sigma|}\omega_\sigma)_{\sigma \in S^p(\mathcal{U})}.$$

Then the following is immediate.

Lemma 5.9. Retain the above notation. Then $\Omega = dI\Omega + Id\Omega$ for all $\Omega \in \Omega^{(m,p)}$ with $m \geq 1$ and $p \geq 0$.

In particular, all vertical complexes are exact.

Now we define maps $K : \Omega^{(m,p)} \to \Omega^{(m,p-1)}$ $(p \geq 1)$ and $K : \Omega^{(m,0)} \to \Omega^m(M)$, playing a role analogous to the maps I above for horizontal complexes. To this end, let $(f_i)_{i \in I}$ be a partition of unity subordinate to $\mathcal{U}$. First, we define $K : \Omega^{(m,0)} \to \Omega^m(M)$ by $(\omega_i)_{i \in I} \mapsto \sum_{i \in I}(f_i\omega_i)$ where $f_i\omega_i$ is extended to the whole of M by zero. Similarly, for $p \geq 1$ and $\Omega = (\omega_\sigma)_{\sigma \in S^p(\mathcal{U})}$ we set $K\Omega := (\zeta_\eta)_{\eta \in S^{p-1}(\mathcal{U})}$ where

$$\zeta_{i_0,\ldots,i_{p-1}} = \sum_{k \in I} f_k \omega_{k,i_0,\ldots,i_{p-1}}$$

in which $f_k\omega_{k,i_0,\ldots,i_{p-1}} = 0$ if $U_k \cap U_{i_0,\ldots,i_{p-1}} = \emptyset$, and if $U_k \cap U_{i_0,\ldots,i_{p-1}} \neq \emptyset$ then

$$f_k\omega_{k,i_0,\ldots,i_{p-1}} := \begin{cases} f_k\omega_{k,i_0,\ldots,i_{p-1}} & \text{on } U_k \cap U_{i_0,\ldots,i_{p-1}} \\ 0 & \text{on } U_{i_0,\ldots,i_{p-1}} \backslash U_k \end{cases}.$$

This is an extension similar to the one in the Mayer-Vietoris sequence.

Lemma 5.10. Retain the above notation. Then $\Omega = \delta K\Omega + K\delta\Omega$ for all $\Omega \in \Omega^{(m,p)}$ with $m \geq 1$ and $p \geq 0$.

In particular, all horizontal complexes are exact. We are now in a position to prove Theorem 5.3, i.e. to show that

$$\mathrm{H}^k_{\mathrm{dR}}(M) \cong \check{\mathrm{H}}^k(\mathcal{U}, \mathbb{R})$$

for every $k \geq 0$ whenever $\mathcal{U}$ is an admissible covering of M. Building on the above, our strategy is as follows. Let $\mathrm{Z}^m_{\mathrm{dR}}(M) = \ker(d : \Omega^m(M) \to \Omega^{m+1}(M))$. Given $\omega \in \mathrm{Z}^m_{\mathrm{dR}}(M)$ we have $d(\delta\omega) = \delta(d\omega) = 0$. Hence there is $\omega_1 \in \Omega^{(m-1,0)}$ with $d\omega_1 = \delta\omega$.

Next, since $d(\delta\omega_1) = \delta(d\omega_1) = \delta^2\omega = 0$, there is $\omega_2 \in \Omega^{(m-2,1)}$ with $d\omega_2 = \delta\omega_1$. Continuing in this fashion we reach $\delta\omega_m \in \Omega^{(0,m)}$ which satisfies $d\delta\omega_m = 0$ and hence $\delta\omega_m \in C^m(\mathcal{U}, \mathbb{R})$. In fact, $\delta(\delta\omega_m) = 0$ and hence $\delta\omega_m \in \mathrm{Z}^m(\mathcal{U}, \mathbb{R}) = \ker(\delta)$.

$$\begin{array}{ccccc} \mathrm{Z}^m_{\mathrm{dR}}(M) & \xrightarrow{\delta} & \Omega^{(m,0)} & & \\ & & \uparrow d & & \\ & & \Omega^{(m-1,0)} & \xrightarrow{\delta} & \Omega^{(m-1,1)} \\ & & & & \uparrow d \\ & & & & \Omega^{(m-2,1)} \end{array}$$

This can be made into a well-defined map from $\mathrm{Z}^m_{\mathrm{dR}}(M) \to \mathrm{Z}^m(\mathcal{U}, \mathbb{R})$ and we will show that it induces the sought-after isomorphism in cohomology. To this end, define for $0 \le n \le m-1$:

$$\mathcal{F}_{m,n} := \{\Omega \in \Omega^{(m-1-n,n)} \mid d\delta\Omega = 0\}$$

and

$$\mathcal{H}_{m,n} := \{\Omega \in \Omega^{(m-1-n,n)} \mid \Omega = X + Y \text{ with } dX = 0 \text{ and } \delta Y = 0\}.$$

Lemma 5.11. Retain the above notation. The map

$$I\delta : \mathrm{Z}^m_{\mathrm{dR}}(M) \xrightarrow{\delta} \Omega^{(m,0)} \xrightarrow{I} \Omega^{(m-1,0)}$$

takes values in $\mathcal{F}_{(m,0)}$ and induces an isomorphism

$$\mathrm{H}^m_{\mathrm{dR}}(M) \to \mathcal{F}_{m,0} \,/\, \mathcal{H}_{m,0}\,.$$

Proof. For $\omega \in \mathrm{Z}^m_{\mathrm{dR}}(M)$ we have $d(\delta\omega) = \delta(d\omega) = 0$ and hence $\delta\omega = dI(\delta\omega)$. From this we deduce $0 = \delta^2\omega = \delta d(I\delta(\omega))$ and hence $I\delta(\omega) \in \mathcal{F}_{m,0}$.

Now, let $\Omega_0 \in \mathcal{F}_{m,0}$. Then $\delta(d\Omega_0) = 0$ and hence $d\Omega_0 \in \Omega^{(m,0)}$ is in the image of δ: Let $\omega \in \Omega^m(M)$ with $\delta\omega = d\Omega_0$. Then we have

$$\delta(d\omega) = d\delta\omega = d^2\Omega_0 = 0$$

and since $\delta : \Omega^{m+1}(M) \to \Omega^{(m+1,0)}$ is injective we get $d\omega = 0$, that is $\omega \in \mathrm{Z}^m_{\mathrm{dR}}(M)$. Thus $dI\delta\omega = d\Omega_0$ and hence $\Omega_0 - I\delta\omega \in \mathcal{H}_{m,0}$. This shows that the composition of $I\delta$ with the projection $\mathrm{Z}^m_{\mathrm{dR}}(M) \to \mathcal{F}_{m,0} \,/\, \mathcal{H}_{m,0}$ is surjective.

Finally, let $\omega = d\alpha$ for some $\alpha \in \Omega^{m-1}(M)$. Then $\delta\omega = \delta d\alpha = d(\delta\alpha)$. Therefore, $d(I\delta\omega) = \delta\omega = d(\delta\alpha)$ and hence $I\delta\omega = \delta\alpha + \beta$ with $d\beta = 0$, i.e. $I\delta\omega \in \mathcal{H}_{m,0}$.

Conversely, if $\omega \in \mathrm{Z}^m_{\mathrm{dR}}(M)$ and $I\delta\omega = \alpha + \beta$ with $d\alpha = 0$ and $\delta\beta = 0$ then $\delta\omega = d(I\delta\omega) = d\beta$. But since $\delta\beta = 0$ there is $\beta' \in \Omega^{m-1}(M)$ with $\delta\beta' = \beta$ and hence $\delta\omega = d\delta\beta' = \delta d\beta'$ whence $\omega = d\beta'$. □

Lemma 5.12. Retain the above notation. The map

$$\delta : \mathcal{F}_{m,m-1} \to \Omega^{(0,m)}$$

takes values in $\mathrm{Z}^m(\mathcal{U}, \mathbb{R})$ and induces an isomorphism

$$\mathcal{F}_{m,m-1} \,/\, \mathcal{H}_{m,m-1} \to \check{\mathrm{H}}^m(\mathcal{U}, \mathbb{R}).$$

Proof. Let $\Omega \in \mathcal{F}_{m,m-1} \subseteq \Omega^{(0,m-1)}$. Then $d\delta\Omega = 0$ and $\delta\Omega = (f_\sigma)_{\sigma \in S^m(\mathcal{U})}$ with f_σ constant on $U_{|\sigma|}$. Thus $\delta\Omega \in C^m(\mathcal{U}, \mathbb{R})$. In addition $\delta\Omega \in \mathrm{Z}^m(\mathcal{U}, \mathbb{R})$ since $\delta^2\Omega = 0$.

Now, let $c \in \mathrm{Z}^m(\mathcal{U}, \mathbb{R})$. Then $dc = 0$ and $\delta c = 0$. Hence there is $\Omega \in \Omega^{(0,m-1)}$ with $\delta\Omega = c$. Clearly, $d\delta\Omega = dc = 0$ and hence $\Omega \in \mathcal{F}_{m,m-1}$. Therefore the map $\delta : \mathcal{F}_{m,m-1} \to \mathrm{Z}^m(\mathcal{U}, \mathbb{R})$ is surjective.

Finally, let $\Omega \in \mathcal{H}_{m,m-1}$, i.e. $\Omega = \alpha + \beta$ with $d\alpha = 0$ and $\delta\beta = 0$. Then $\delta\Omega = \delta\alpha$ and since $d\alpha = 0$ we have $\alpha \in \check{\mathrm{C}}^{m-1}(\mathcal{U}, \mathbb{R})$. Hence $\delta\Omega = \delta\alpha \in \check{\mathrm{C}}^m(\mathcal{U}, \mathbb{R})$. Conversely, assume that $\delta\Omega = \delta\gamma$ with $\gamma \in \check{\mathrm{C}}^{m-1}(\mathcal{U}, \mathbb{R})$. Then $d\gamma = 0$ and $\delta(\Omega - \gamma) = 0$ and therefore $\Omega \in \mathcal{H}_{m,m-1}$. □

Now, for $0 \leq n \leq m-2$ consider the following diagram:

$$\begin{array}{ccc} \Omega^{(m-1-n,n)} & \xrightarrow{\delta} & \Omega^{(m-1-n,n+1)} \\ & & \uparrow{\scriptstyle d} \\ & & \Omega^{(m-1-(n+1),n+1)}. \end{array}$$

Lemma 5.13. Retain the above notation. Then the map

$$\delta : \mathcal{F}_{m,n} \to \Omega^{(m-1-(n+1)),n+1)}$$

takes values in $\mathcal{F}_{m,n+1}$ and induces an isomorphism

$$\mathcal{F}_{m,n} \,/\, \mathcal{H}_{m,n} \to \mathcal{F}_{m,n+1} \,/\, \mathcal{H}_{m,n+1} \,.$$

The proof of Lemma 5.13 is similar to the proofs of Lemmas 5.11 and 5.12 and is left to the reader. Overall, this proves Theorem 5.3.

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