Exercise Sheet 1

1. Let $\Gamma = (V, E)$ be a finite graph of degree ≤ 6 with no loops and no multiple edges. In the lecture we have seen a theorem due to Barzdin and Kolmogorov which states that Γ can be embedded into $\mathbb{R}^3$ in a "thickened way". Try to prove this.

Note: The statement of Barzdin and Kolmogorov which we have seen in the lecture is actually more precise, as it also states how much space one needs at most to embed Γ into $\mathbb{R}^3$. However, you do not need to be so precise.

2. Compute the Cheeger constant of the following graphs:

a) The complete graph with n vertices.

b) The linear graph with n vertices.

3. Let $k \geq 2$ be an integer, $G = \mathfrak{S}_k$ the symmetric group on k letters. Let H be a subgroup of G such that: (i) H acts transitively on $\{1, \ldots, k\}$; (ii) H contains at least one transposition; (iii) H contains a cycle of length $p > k/2$ such that p is prime. The goal is to prove that, in fact, we have $H = G$.

For this, let $\Gamma = (V, E)$ be the simple graph with $V = \{1, \ldots, k\}$ and with an edge between any pair $(i, j) \in V \times V$ such that $i \neq j$ and the transposition $(i\ j)$ is in H. Assumption (ii) means that the edge set is not empty.

a) Show that any connected component in Γ is a complete graph.

b) Show that it is enough to show that Γ is connected in order to prove that $H = G$.

c) Show that the action of H on $\{1, \ldots, k\}$ induces an action of H on Γ by automorphisms. Show then that H acts transitively on the set of all connected components of Γ. Deduce that all such components are isomorphic.

d) Show that a p-cycle $\sigma \in H$ as in (iii) must fix (globally, not necessarily pointwise) each component of Γ, and conclude from this.

4. a) Let $\Gamma = (V, E)$ be a connected bipartite graph with a bipartite decomposition $V = V_0 \cup V_1$. If $x_0 \in V_0$, show that

$$V_0 = \{y \in V \mid \text{there is a path of even length joining } x_0 \text{ to } y\}. \tag{1}$$

Bitte wenden!

b) Deduce that the partition of edges $V = V_0 \cup V_1$ which exhibits the bipartiteness of a connected bipartite graph is unique, i.e., if $W_0 \cup W_1$ is another such partition, we have $(W_0, W_1) = (V_0, V_1)$ or $(W_0, W_1) = (V_1, V_0)$.

c) Let Γ be an arbitrary connected graph, and let W be the right-hand side in (1). What is W when Γ is not bipartite?

d) Show that a forest is always bipartite.

e) Show that if Γ is finite and not bipartite, its girth is finite. In fact, show that $\text{girth}(\Gamma) \leq 2\ \text{diam}(\Gamma) + 1$, and that this is best possible.

Submission: Wednesday, 9th March 2016 during the exercise class.