Exercise Sheet 10

In this sheet, we will establish a probabilistic-method variant of a sieve, which is known as Lovász Local Lemma. There are many applications of Local Lemma in mathematics. In this sheet, we illustrate Local Lemma to a coloring problem for hypergraphs, which are the following generalization of graphs. Let $r \geq 2$. A *simple r-uniform hypergraph H* is a pair of two sets (V, E), where V is the set of vertices, and every $e \in E$ is a subset of V of size exactly r. Note that the case $r = 2$ is equivalent to the standard notion of undirected simple graphs.

Lemma *[Lovász Local Lemma] Let $A_1, A_2, \dots, A_n$ be events in an arbitrary probability space. Suppose that each event A_i is mutually independent of a set $\mathcal{A}_i$ that contain all the other events A_j but at most d (i.e., $|\mathcal{A}_i| \geq n - d - 1$), and that $P[A_i] \leq p$ for all $1 \leq i \leq n$. If $4pd \leq 1$, then*

$$P\left[\bigwedge_{1 \leq i \leq n} \overline{A_i}\right] > 0.$$

1. A joke: take a random function f from a set of size n to a set of size m. Use a simple union-bound that if $m > 10n^2$, then with positive probability f is injective.

2. Indeed the previous exercise was quite an ineffective way for showing an existence of an injective function, and the bound on m is really horrible. Nevertheless, let's try it again and with Local Lemma. Prove that if $m > 10n$, then with positive probability f is injective.

3. In the previous two tasks, we used the probabilistic method to show that there exists an injective function from N to M whenever $|M| \geq 10|N|$. This is clearly a stupid toy problem, and we can do better without probabilistic method. But we have seen that Local Lemma can make a significant improvement over just using a union-bound. So let us apply it to something we do not know how to prove without probabilistic method.

Let $H = (V, E)$ be an r-uniform hypergraph such that every edge $e \in E$ intersects at most 2^{r-3} other hyperedges. Prove that there exist a partition of the vertex-set V into two parts R (red) and B (blue) such that no hyperedge $e \in E$ is monochromatic, i.e., completely contained in R or B. In combinatorial words, the hypergraph H is called 2-colorable.

4. Ok, in the rest of the sheet, we prove the Local Lemma. As the first step, show that the Lemma follows from the following more general statement (also known as the general version of Local Lemma):

Bitte wenden!

Lemma *Let $A_1, A_2, \ldots, A_n$ be events in an arbitrary probability space. Let D be a (dependency) graph with vertices $A_1, \ldots, A_n$ such that that each event A_i is mutually independent of a set of all the events that are non-neighbors of A_i in D. If there exist non-negative numbers $x_1, \ldots, x_n$ such that $x_i < 1$ for every $i \in [n]$ and $P[A_i] \leq x_i \cdot \Pi_{ij \in E(G)}(1 - x_j)$, then*

$$P\left[\bigwedge_{1 \leq i \leq n} \overline{A_i}\right] \geq \prod_{1 \leq i \leq n} (1 - x_i).$$

5. Establish, by induction on s, the following claim, and use it to prove the general local lemma.

Claim *For any $S \subset \{A_1, \ldots, A_n\}$, where $|S| = s$, and $i \notin S$, the following is true:*

$$P\left[A_i | \bigwedge_{j \in S} \overline{A_j}\right] \leq x_i.$$

Submission: On ∞ June 2016, since there will be no other exercise class.