Exercise Sheet 2

1. a) Let $G = (V, E)$ be a finite connected graph on n vertices with no loops and no multiple edges. Show that the Cheeger constant $h(G) \geq \frac{2}{n}$.

b) For infinitely many integers n, construct n-vertex graphs G with the diameter bounded by a constant such that $h(G) \leq \frac{C}{n}$, where $C > 0$ is an absolute constant independent on n.

c) For any integer $d \geq 3$ and infinitely many integers n, construct n-vertex d-regular graphs G' with the diameter bounded by a $C' \cdot \log(n)$ and $h(G') \leq \frac{C'}{n}$, where $C' > 0$ is again some constant independent on n.

2. Let G be a finite d-regular graph on n vertices with girth $g \geq 3$. Show that

$$n \geq d(d-1)^{\lfloor (g-3)/2 \rfloor}.$$

3. Let G be a finite graph on n vertices with minimum degree at least 3. Show that the girth of G is at most $O(\log(n))$.

4. For any integer $d \geq 3$ and any integer $g \geq 3$, construct a finite d-regular graph with girth at least g.

Hint: For a fixed d, construct a d-regular graph $G' = (V', E')$ with girth g from a d-regular graph $G = (V, E)$ with smaller girth in the following way:

- *Let $\mathcal{F}$ be the set of all functions $f : E \to \{0, 1\}$.*
- *Set $V' := V \times \mathcal{F}$. Put an edge in G' between two vertices (v_1, f_1) and (v_2, f_2) if and only if $\{v_1, v_2\}$ is an edge in G, and the functions f_1 and f_2 agree on all the edges except $\{v_1, v_2\}$.*

5. Let G_n be the 3-regular Cayley graph of the symmetric group S_n with respect to the generating set $\{\tau_{12}, (1\ 2\ \cdots\ n), (1\ 2\ \cdots\ n)^{-1}\}$.

a) Show that the diameter of G_n is at most $C \cdot n^2$ for some constant $C > 0$.

b) Show that the Cheeger constant $h(G_n)$ is at most $\frac{C'}{n}$ for a fixed constant $C' > 0$.

$\star$) Prove as good lower bound on the diameter of G_n as you will be able to.

Submission: Wednesday, 16th March 2016 during the exercise class.