Exercise Sheet 3

1. The goal of this exercise is to give an alternative proof of $\|M\| \leq 1$ where M denotes the Markov averaging operator on a finite graph Γ.

a) Explain why the norm of M is the maximum of the absolute values of its eigenvalues.

b) If λ is an eigenvalue, show directly that $|\lambda| \leq 1$.
Hint: Use the maximum norm instead of the L^2-norm.

2. This exercise discusses the "geometric" computation of $\ker(M-1)$.

a) Show that if φ is the characteristic function of a connected component of Γ, we have

$$M\varphi = \varphi.$$

b) Show that, in order to prove that these characteristic functions span $\ker(M-1)$, it is enough to prove that a real-valued element of $\ker(M-1)$ is constant on each connected component of Γ.

c) Let $W \subset V$ be a connected component. Let φ be a real-valued element of $\ker(M-1)$, let m be the maximum value of $\varphi(x)$ on W, and let $x_0 \in W$ a vertex where $\varphi(x_0) = m$. Show that $\varphi(x) = m$ for all x connected to x_0 by at least one edge.

d) Deduce that φ is equal to m on all of W and conclude.

e) Using similar methods, determine $\ker(M+1)$.

3. Let n be a positive integer and let $J_1, \ldots, J_{n-1}$ be independent random integers, where J_k is uniform on $\{k, k+1, \ldots, n\}$. Define elements $\sigma_0, \sigma_1, \ldots, \sigma_{n-1} \in \mathcal{S}_n$ recursively as follows: σ_0 is the identity permutation and for $k \in \{1, \ldots, n-1\}$, σ_k is given by

$$\sigma_k(i) = \begin{cases} \sigma_{k-1}(i) & \text{if } i \neq J_k, i \neq k, \\ \sigma_{k-1}(J_k) & \text{if } i = k, \\ \sigma_{k-1}(k) & \text{if } i = J_k. \end{cases}$$

Show that σ_{n-1} is uniformly distributed on $\mathcal{S}_n$.

Bitte wenden!

4. Recall that a random walk $(x_n)_{n\geq 0}$ on a graph Γ is said to be recurrent if almost surely it visits every vertex x in Γ infinitely many times, i.e., if

$$\mathbb{P}[x_n = x \text{ for inifinitely many } n] = 1.$$

If this is not the case then we say the walk is transient.

a) Show that the random walk on the Cayley graph $\mathcal{C}\big(\mathbb{Z}, \{-1, 1\}\big)$ is recurrent.

b) Show that the random walk on the Cayley graph

$$\mathcal{C}\big(\mathbb{Z}^2, \{(-1,0), (1,0), (0,-1), (0,1)\}\big)$$

is recurrent.

c) Show that the random walk on the Cayley graph

$$\mathcal{C}\big(\mathbb{Z}^3, \{(-1,0,0), (1,0,0), (0,-1,0), (0,1,0), (0,0,-1), (0,0,1)\}\big)$$

is transient.

Submission: Wednesday, 23th March 2016 during the exercise class.