Exercise Sheet 4

1. Let $G = (V, E)$ be a finite simple graph and for every vertex $v \in V$, let t_v be the expected time that a simple random walk starting at v takes in order to visit every vertex from V at least once. The cover time of G is then the maximum of t_v taken over all $v \in V$.

a) What is the cover time of the complete graph on n vertices K_n?

b) What is the cover time of the n-vertex cycle C_n?

$\star$) For every n, try to find an n-vertex graph with cover time of order $\Theta(n^3)$.

2. Let T be the infinite 3-regular tree. Show that a simple random walk on T is transient, i.e., probability that it visits each vertex infinitely many times is strictly less than one.

3. Let G be a d-regular graph with the adjacency matrix A so that every eigenvalue of A apart from the largest one (which is equal to d) is *in the absolute value* smaller than λ.

a) For two sets $S, T \subseteq V$, let $e(S, T)$ be the number of ordered pairs of vertices $(s, t) \in S \times T$ such that $\{s, t\}$ is an edge of G. Prove that

$$\left| e(S,T) - \frac{d \cdot |S| \cdot |T|}{n} \right| \leq \lambda \cdot \sqrt{|S| \cdot |T|}.$$

b) A set of vertices $I \subseteq V$ is called independent in G if the subgraph of G induced by I does not contain any edge. Show that $|I| \leq \frac{\lambda}{d} \cdot n$ for any independent set I of G.

c) Observe that there is an independent set I in G such that $|I| \geq \frac{n}{d}$.

Submission: Wednesday, 30th March 2016 during the exercise class.