Exercise Sheet 5

1. Let $\Gamma = (V, E)$ be a finite connected graph with $|V| \geq 2$. Let $0 = \lambda_0 \leq \lambda_1 \leq \ldots \leq \lambda_{n-1} \leq 2$ be the eigenvalues of Id $- M$ where M denotes the Markov operator. Since Γ is connected, $\lambda_1 \neq 0$. λ_1 is called the spectral gap of Γ and from the lecture we know that

$$\frac{v_-^2}{2v_+}\lambda_1 \leq h(\Gamma) \leq v_+\sqrt{2\lambda_1}$$

where $v_- = \min_x \deg(x)$ and $v_+ = \max_x \deg(x)$. This tells us that the expansion in the graph Γ is small if and only if λ_1 is small.

Now, let $\Gamma = (V, E)$ be a finite graph which is not necessarily connected. The goal of this exercise is to derive similar inequalities, which tell us that Γ is close to having a bipartite connected component if and only if $2 - \lambda_{n-1}$ is small. Moreover, $2 - \lambda_{n-1} = 0$ if and only if Γ has a bipartite connected component. For this, we first need to define a quantity which measures how far away Γ is from having a bipartite connected component. This quantity is $\beta(\Gamma)$ which is defined by

$$\beta(\Gamma) = \min_{y \in \{-1,0,1\}^V \setminus \{\mathbf{0}\}} \frac{\sum_{\{u,v\}\in E} |y_u + y_v|}{\sum_{v \in V} \deg(v)|y_v|}.$$

In this exercise, we will prove the inequalities

$$\frac{2 - \lambda_{n-1}}{2} \leq \beta(\Gamma) \leq \sqrt{2(2 - \lambda_{n-1})}.$$

a) Explain that $\beta(\Gamma)$ is small when Γ is "close" to having a bipartite connected component and check that $\beta(\Gamma) = 0$ if and only if Γ has a bipartite connected component.

b) Show that

$$2 - \lambda_{n-1} = \min_{x \in \mathbb{R}^n \setminus \{0\}} \frac{\sum_{\{u,v\}\in E} (x_u + x_v)^2}{\sum_{v \in V} \deg(v) x_v^2}$$

and that every minimizer is an eigenvector of λ_{n-1}.

c) Prove the first inequality, i.e., prove that

$$2 - \lambda_{n-1} \leq 2\beta(\Gamma).$$

Bitte wenden!

d) Let $\mathbf{x} \in [-1,1]^V \setminus \{\mathbf{0}\}$ and choose $t \in]0,1]$ at random such that t^2 is uniformly distributed in $[0,1]$. Define $\mathbf{y}^t \in \{-1,0,1\}^V$ by

$$y_v^t = \begin{cases} -1 & \text{if } x_v \leq -t, \\ 0 & \text{if } -t < x_v < t, \\ 1 & \text{if } x_v \geq t, \end{cases}$$

for every $v \in V$.

Show that if x_u and x_v have the same sign, then

$$\mathbb{E}\big[|y_u^t + y_v^t|\big] = x_u^2 + x_v^2$$

and if x_u and x_v have opposite signs, then

$$\mathbb{E}\big[|y_u^t + y_v^t|\big] = |x_u^2 - x_v^2|.$$

Conclude that in both cases the inequality

$$\mathbb{E}\big[|y_u^t + y_v^t|\big] \leq |x_u + x_v|\big(|x_u| + |x_v|\big)$$

holds.

e) Let $\mathbf{x}$, t and $\mathbf{y}^t$ be as in part d). Prove that

$$\mathbb{E}\left[\sum_{\{u,v\}\in E} |y_u^t + y_v^t|\right] \leq \sqrt{\sum_{\{u,v\}\in E} (x_u + x_v)^2}\sqrt{\sum_{\{u,v\}\in E} (|x_u| + |x_v|)^2},$$

$$\mathbb{E}\left[\sum_{v\in V} \deg(v)|y_v^t|\right] = \sum_{v\in V} \deg(v)x_v^2$$

and

$$\sum_{\{u,v\}\in E} \big(|x_u| + |x_v|\big)^2 \leq 2\sum_{v\in V} \deg(v)x_v^2.$$

Conclude that

$$\frac{\mathbb{E}\left[\sum_{\{u,v\}\in E} |y_u^t + y_v^t|\right]}{\mathbb{E}\left[\sum_{v\in V} \deg(v)|y_v^t|\right]} \leq \sqrt{2\frac{\sum_{\{u,v\}\in E}(x_u + x_v)^2}{\sum_{v\in V} \deg(v)x_v^2}}.$$

f) Let X and Y be random variables such that $\mathbb{P}[Y > 0] = 1$. Show that

$$\mathbb{P}\left[\frac{X}{Y} \leq \frac{\mathbb{E}[X]}{\mathbb{E}[Y]}\right] > 0.$$

g) Finally, show that

$$\beta(\Gamma) \leq \sqrt{2(2 - \lambda_{n-1})}.$$

Submission: Wednesday, 13th April 2016 during the exercise class.