Exercise Sheet 6

The main theme of this sheet is to introduce the so-called *Zig-Zag product*, which will give us an explicit combinatorial construction of expander graphs. For the whole sheet, G will be a "large" D-regular graph on N vertices, and H will be a "small" d-regular graph on D vertices. Furthermore, we assume that the neighbors of each vertex of G have a prescribed order. So for instance, if a vertex $v \in V(G)$ has neighbors $u_1, \ldots, u_D$, we will use the *i-th neighbor* of v to denote the vertex u_i. Note that if u and v are two adjacent vertices in G, it might be that u is the i-th neighbor of v, and v is the j-th neighbor of u for $i \neq j$. We will also assume that the vertex-set of H is $\{1, 2, \ldots, D\}$.

1. Before introducing the Zig-Zag product itself, let us start with a simpler notion of a product of graphs G and H called the *replacement product*. We define the result of the replacement product $G \textcircled{r} H$ to be the following $(d+1)$-regular graph with $N \cdot D$ vertices:

- The vertex set will be $V(G) \times V(H)$.
- For any vertex $u \in V(G)$, the vertices (u, i) and (u, j) of $G \textcircled{r} H$ where $\{i, j\} \subseteq V(H)$, will form an edge if and only if $\{i, j\}$ is an edge of H.
- Finally, two vertices (u, i) and (v, j) for $u \neq v$ will be connected by an edge if and only if $\{u, v\}$ is an edge of G, v is the i-th neighbor of u, and u is the j-th neighbor of v.

In other words, we expand every vertex of $V(G)$ into D vertices (each such expansion is called *cloud*), each original edge $\{u, v\}$ of G will become an edge between (u, i) and (v, j), where i is the integer denoting the position of v among neighbors of u, analogously u is the j-th neighbor of v, and finally inside each cloud we place a copy of H.

a) Determine the graph $K_{d+1} \textcircled{r} K_d$.

b) Show that if G is connected and H is connected, then $G \textcircled{r} H$ will also be a connected graph.

2. Now we introduce an analogous product that keeps not only the connectivity, but also good expansion. The product is called Zig-Zag product, denoted by $G \textcircled{z} H$. The vertex-set of $G \textcircled{z} H$ will be again $V(G) \times V(H)$, and the edges will correspond to the following paths of length 3 in $G \textcircled{r} H$. Two vertices (u, i) and (v, ℓ) of $G \textcircled{z} H$ will be connected by an edge if and only if there exist $j \in V(H)$ and $k \in V(H)$ such that:

- (u, i) is connected to (u, j) in $G \textcircled{r} H$ (i.e., $\{i, j\}$ is an edge of H),
- (u, j) is connected to (v, k) in $G \textcircled{r} H$ (i.e., v is the j-th neighbor of u, and u is the k-th neigbor of v in G), and
- (v, k) is connected to (v, ℓ) in $G \textcircled{r} H$ (i.e., $\{k, \ell\}$ is an edge of H).

Loosely speaking, the edges of $G \textcircled{z} H$ exactly correspond to "zig-zag" paths between two clouds in $G \textcircled{z} H$.

a) Show that the $G \textcircled{z} H$ is a d^2-regular graph.

b) What is the graph $K_{d+1} \textcircled{z} K_d$?

c) Again, if G is connected and H is connected, show that $G \textcircled{z} H$ is also connected.

3. Ok, we introduced the required notion, now it is time for a serious stuff. Prove that if $\lambda_{\max}(G)$, which is the second largest eigenvalue in absolute value of the adjacency matrix of G (i.e., $\lambda_{\max} = \max\{\lambda_2(A(G)), -\lambda_N(A(G))\}$ for $A(G)$ the adjacency matrix of G) is equal to $\alpha \cdot D$, and analogously $\lambda_{\max}(H) = \beta \cdot d$, then

$$\lambda_{\max}(G \textcircled{z} H) \leq d^2 \cdot \left(\alpha + \beta + \beta^2\right) .$$

Hint: Decompose the adjacency matrix A of $G \textcircled{z} H$ into three pieces using the interpretation of the edges in $G \textcircled{z} H$ as paths $G \textcircled{r} H$ which go inside a first cloud following edges of H, then between two clouds following edges of G, and finally inside the second cloud again following edges of H. Then recall the Rayleigh formula and the spectral norm of a matrix, and use them to bound the $\lambda_{\max}(G \textcircled{z} H)$.

4. So if G and H are good expanders, we showed that the Zig-Zag product outputs also an expander. The final step is to appropriately use the product and construct a sequence of bounded-degree expanders. Suppose that H is a d-regular graph on d^4 vertices with $\lambda_{\max}(H) = d/4$ (such a graph exists, either use the probabilistic method to show that, or take it as granted; if needed, you could brute-force all d-regular graphs with d^4 vertices in order to find this one graph).

Let $G_0 := H$, $G_1 := (G_0)^2$ and for $k \geq 2$, let $G_k := (G_{k-1})^2 \textcircled{z} H$. The square of a graph G, denoted by G^2, is the graph on the same set of vertices as G, where the edges correspond to walks of length two in G (so in particular, if G is a d-regular graph without loops, G^2 is a d^2-regular graph containing exactly d loops on each vertex).

Show that for all $k \geq 1$, G_k is d^2-regular graph with d^{4k} vertices and $\lambda_{\max}(G_k) \leq d^2/2$.

Submission: Wednesday, 20th April 2016 during the exercise class.