Exercise Sheet 8

In this sheet, we focus on a different group property then Kazhdan's property (T) that yields expansion in the corresponding Cayley graphs. Let G be a finite group and let $d(G)$ be the minimal dimension of a non-trivial unitary representation of G.

1. Let $p \neq 3$ be an odd prime, and let

$$S := \left\langle \begin{pmatrix} 1 & \pm 3 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \pm 3 & 1 \end{pmatrix} \right\rangle \subset \mathbf{SL}_2(\mathbb{F}_p).$$

Show that the girth of the Cayley graph corresponding to S has girth of order $\Omega(\log p)$.

2. For a finite field $\mathbb{F}_p$, the group $\mathbf{SL}_2(\mathbb{F}_p)$ has no non-trivial unitary representation of dimension smaller than $(p-1)/2$.

3. Let G be a finite group. If $A \subseteq G$ such that $|A| > |G|/2$, then $A \cdot A = G$.

4. Let G be a finite group. If $A \subseteq G$ such that $|A| > (|G|/d(G))^{1/3}$, then $A \cdot A \cdot A = G$.

Submission: Wednesday, 11th May 2016 during the exercise class.