Exercise Sheet 9

Let $n > 2$ and define

$$T_1 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \in M_2(\mathbb{Z}/n\mathbb{Z}), \qquad T_2 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \in M_2(\mathbb{Z}/n\mathbb{Z}),$$
$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in (\mathbb{Z}/n\mathbb{Z})^2, \qquad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in (\mathbb{Z}/n\mathbb{Z})^2.$$

Define a partial order on $(\mathbb{Z}/n\mathbb{Z})^2$ as follows: Let $(x_1, x_2), (y_1, y_2) \in (\mathbb{Z}/n\mathbb{Z})^2$. Let x_1', x_2', y_1', y_2' be integers in the interval $[-n/2, n/2)$ such that $x_1' \equiv x_1 \mod n$, $x_2' \equiv x_2 \mod n$, $y_1' \equiv y_1 \mod n$ and $y_2' \equiv y_2 \mod n$. We say that $(x_1, x_2) > (y_1, y_2)$ if $|x_1'| \geq |y_1'|$ and $|x_2'| \geq |y_2'|$ and at least one of the inequalities is strong.

Define the "diamond" $D \subset (\mathbb{Z}/n\mathbb{Z})^2$ by

$$D = \left\{(x_1, x_2) \in (\mathbb{Z}/n\mathbb{Z})^2 \middle| |x_1'| + |x_2'| < n/2\right\}$$

where x_1' and x_2' are defined as before.

Furthermore, define

$$\gamma \colon (\mathbb{Z}/n\mathbb{Z})^2 \times (\mathbb{Z}/n\mathbb{Z})^2 \longrightarrow \mathbb{R}$$

by

$$\gamma(x, y) = \begin{cases} \frac{5}{4} & \text{if } x > y, \\ \frac{4}{5} & \text{if } x < y, \\ 1 & \text{otherwise.} \end{cases}$$

1. a) Show that for every $z = (z_1, z_2) \in D$, either

i) three of the four points $T_1 z$, $T_2 z$, $T_1^{-1} z$ and $T_2^{-1} z$ are $> z$ and one is $< z$ or

ii) two of the four points $T_1 z$, $T_2 z$, $T_1^{-1} z$ and $T_2^{-1} z$ are $> z$ and two are incomparable with z.

b) Prove that there exists some absolute constant $\kappa < 4$ (i.e. κ does not depend on n) such that for every $z \in D$,

$$\gamma(z, T_1 z) + \gamma(z, T_1^{-1} z) + \gamma(z, T_2 z) + \gamma(z, T_2^{-1} z) \leq \kappa.$$

c) Show that there exists an absolute constant $\kappa < 4$ such that for all $z \in (\mathbb{Z}/n\mathbb{Z})^2$ with $z \neq (0,0)$,

$$|\cos(\pi z_1/n)|\big(\gamma(z, T_2 z)+\gamma(z, T_2^{-1} z)\big)+|\cos(\pi z_2/n)|\big(\gamma(z, T_1 z)+\gamma(z, T_1^{-1} z)\big) \leq \kappa.$$

d) Prove that there exists an absolute constant $\kappa < 4$ such that for every non-negative function $g\colon (\mathbb{Z}/n\mathbb{Z})^2 \longrightarrow \mathbb{R}$ with $g(0) = 0$,

$$\sum_{z\in(\mathbb{Z}/n\mathbb{Z})^2} g^2(z)\Big(|\cos(\pi z_1/n)|\big(\gamma(z, T_2 z) + \gamma(z, T_2^{-1} z)\big) + |\cos(\pi z_2/n)|\big(\gamma(z, T_1 z) + \gamma(z, T_1^{-1} z)\big)\Big) \leq \kappa \sum_{z\in(\mathbb{Z}/n\mathbb{Z})^2} g^2(z)$$

e) Show that there exists an absolute constant $\kappa < 4$ such that for every non-negative function $g\colon (\mathbb{Z}/n\mathbb{Z})^2 \longrightarrow \mathbb{R}$ with $g(0) = 0$,

$$\sum_{z\in(\mathbb{Z}/n\mathbb{Z})^2} 2g(z)\big(g(T_2^{-1}z)|\cos(\pi z_1/n)|+g(T_1^{-1}z)|\cos(\pi z_2/n)|\big) \leq \kappa \sum_{z\in(\mathbb{Z}/n\mathbb{Z})^2} g^2(z)$$

Hint: Use the inequality $2\alpha\beta \leq \gamma\alpha^2 + \gamma^{-1}\beta^2$ which is valid for all $\alpha, \beta \geq 0$ and $\gamma > 0$, use that $\gamma(x, y)\gamma(y, x) = 1$ and exploit the fact that z_2 is invariant under T_1 and likewise with z_1 and T_2.

f) Prove that there exists an absolute constant $\kappa < 4$ such that for every function $h\colon (\mathbb{Z}/n\mathbb{Z}) \longrightarrow \mathbb{C}$ with $h(0) = 0$,

$$\Bigg|\sum_{z\in(\mathbb{Z}/n\mathbb{Z})^2} \overline{h(z)}\Big(h(T_2^{-1}z)\big(1 + e(-z_1/n)\big) + h(T_1^{-1}z)\big(1 + e(-z_2/n)\big)\Big)\Bigg| \leq \kappa \sum_{z\in(\mathbb{Z}/n\mathbb{Z})^2} |h(z)|^2.$$

2. Let $f\colon (\mathbb{Z}/n\mathbb{Z})^2 \longrightarrow \mathbb{C}$ be a function and recall the definition of the Fourier transform

$$\hat{f}(x) = \sum_{b\in(\mathbb{Z}/n\mathbb{Z})^2} \overline{f(b)}e\Big(\frac{\langle b, x\rangle}{n}\Big).$$

Recall that for all functions $f, g\colon (\mathbb{Z}/n\mathbb{Z})^2 \longrightarrow \mathbb{C}$, we have

$$\langle f, g\rangle = \langle \hat{f}, \hat{g}\rangle$$

and that if A is a nonsingular 2×2 matrix over $\mathbb{Z}/n\mathbb{Z}$, $b \in (\mathbb{Z}/n\mathbb{Z})^2$ and $g(x) = f(Ax + b)$, then

$$\hat{g}(y) = e\Big(\frac{-\langle A^{-1}b, y\rangle}{n}\Big)\hat{f}\big((A^{-1})^T y\big).$$

Siehe nächstes Blatt!

Show that there exists an absolute constant $\kappa < 8$ such that for every function $f\colon (\mathbb{Z}/n\mathbb{Z})^2 \longrightarrow \mathbb{R}$ satisfying $\sum_{z\in(\mathbb{Z}/n\mathbb{Z})^2} f(z) = 0$, the following inequality holds

$$2 \sum_{z\in(\mathbb{Z}/n\mathbb{Z})^2} f(z)\big(f(T_1 z) + f(T_1 z + e_1) + f(T_2 z) + f(T_2 z + e_2)\big) \leq \kappa \sum_{z\in(\mathbb{Z}/n\mathbb{Z})^2} f^2(z)\big)$$

3. We define the following family of 8-regular graphs $(\Gamma_n)_{n\geq 3}$: The vertex set of Γ_n is $V_n = (\mathbb{Z}/n\mathbb{Z})^2$ and every vertex $v = (v_1, v_2)$ is adjacent to the four vertices $T_1 v, T_2 v, T_1 v + e_1, T_2 v + e_2$ and the other four neighbours of v are obtained by the four inverse transformations. Note that all calculations are modulo n and that this is an undirected 8-regular graph (that may have multiple edges and self loops). The goal is to show that $(\Gamma_n)_{n\geq 3}$ is an expander family. For this, it is enough to show that there exists an absolute constant $c > 0$ such that $\lambda_1(\Gamma_n) > c$ for all $n \geq 3$. First of all, recall that

$$\lambda_1(\Gamma) = \min_{\substack{\varphi\in L^2(\Gamma)\\ \langle\varphi,1\rangle=0}} \frac{\langle \Delta_{\Gamma_n}\varphi, \varphi\rangle}{\langle \varphi, \varphi\rangle},$$

where $\Delta_{\Gamma_n} = \mathrm{Id} - M$ denotes the Laplace operator and M the Markov operator.

a) Using exercise 2, show that there exists an absolute constant $\kappa < 1$ such that for every $\varphi \in L^2(\Gamma_n)$ with $\langle \varphi, 1\rangle = 0$,

$$\langle M\varphi, \varphi\rangle < \kappa\langle \varphi, \varphi\rangle.$$

b) Show that there exists an absolute constant $c > 0$ such that $\lambda_1(\Gamma_n) > c$ for all $n \geq 3$.

Submission: Wednesday, 25th May 2016 during the exercise class.