Solution 1

1. We provide no solution for this exercise.

2. a) Let $\Gamma = (V, E)$ be a complete graph with n vertices. Then, for every $W \subset V$ with $0 < |W| \leq \frac{|V|}{2}$ the following holds: If we denote $m = |W|$, we see that

$$|\partial W| = |W||V \setminus W| = k(n-k).$$

Hence

$$\frac{|\partial W|}{|W|} = \frac{k(n-k)}{k} = n - k.$$

So, the Cheeger constant is

$$h(\Gamma) = \min_{\substack{W \subset V \\ 0<|W|\leq \frac{|V|}{2}}} \frac{|\partial W|}{|W|} = \min_{\substack{W \subset V \\ 0<|W|\leq \frac{|V|}{2}}} n - k = \left\lfloor \frac{n}{2} \right\rfloor.$$

b) Let $\Gamma = (V, E)$ be a linear graph with n vertices. We know that $|\partial W| \geq 1$ for all $W \subset V$ with $0 < |W| \leq \frac{|V|}{2}$.

3. a) It is enough to show that whenever two vertices a and b are connected by a path, then there exists an edge connecting a and b. So, lets assume there exists a path of length k between a and b. We denote the $k+1$ vertices along the path by $i_0, \dots, i_k$, with $i_0 = a$ and $i_k = b$. By the definition of Γ, this means that H contains the transpositions $(i_\ell \, i_{\ell+1})$ for $\ell \in \{0, \dots, k-1\}$. We need to show that H then also contains the transposition $(i_0 \, i_k)$, which corresponds to an edge from $a = i_0$ to $b = i_k$. But since

$$(i_0 \, i_k) = (i_{k-1} \, i_k)(i_{k-2} \, i_{k-1}) \dots (i_1 \, i_2)(i_0 \, i_1)(i_1 \, i_2) \dots (i_{k-2} i_{k-1})(i_{k-1} \, i_k) \in H,$$

we are done.

b) If Γ is connected, then by part a), Γ is the complete graph. Hence H contains all possible transpositions of the elements in $\{1, \dots, k\}$. It is a well known fact that the transpositions generate the symmetric group $\mathfrak{S}_k$, which implies that $H = G$.

c) Let $h \in H$. Then a vertex $i \in V$ is mapped to $h \cdot i$. Furthermore, an edge (i, j) is mapped to an edge $(h \cdot i, h \cdot j)$. First we have to check that map is well-defined, i.e. we have to check that $(h \cdot i, h \cdot j)$ is an edge of Γ. But this is clear since the transposition

$$(h \cdot i \; h \cdot j) = h(i \; j)h^{-1}$$

is contained in H. The bijectivity is now clear, since h simply permutes of the vertices. Hence H acts on Γ by automorphisms.

We also have to show that H acts transitively on the set of connected components of Γ. This follows immediately from assumption (i) on H. For two connected components, select a vertex in each component, say a and b. Then since H acts transitively on $\{1, \ldots, k\}$, there exists $h \in H$ such that $h \cdot a = b$, i.e., h corresponds to an automorphism of Γ which maps a to b and hence it maps the connected component containing a to the connected component containing b.

Since the action of h is by automorphisms, it is therefore also clear that all connected components of Γ are isomorphic.

d) Let $\sigma \in H$ be a p-cycle of length $p > k/2$. First note that since by assumption (ii) there exists at least one edge in Γ and hence there exists a connected component in Γ with at least two vertices. Since by part c) all connected components are isomorphic, every connected component of Γ contains at least two vertices. We denote the number of vertices which each connected component contains by $m \geq 2$.

Now, suppose by contradiction, the action of $\sigma = (i_0 \; \ldots \; i_{p-1})$ on Γ does not fix the connected components of Γ. This means that the vertices $i_0, \ldots, i_{p-1}$ are not all contained in the same connected component of Γ. Say the p vertices $i_0, \ldots, i_{p-1}$ are contained the u connected components $V_1, \ldots, V_u$. Since $|V_1 \cup \ldots V_u| = um$ with $u, m \geq 2$, we know that $p < um$ since p is prime. Hence there exists $a \in V_0 \cup \ldots V_u$ with $a \notin \{i_0, \ldots, i_{p-1}\}$. Without loss of generality, we can assume that $a, i_0 \in V_1$. We now show that then also $i_1, \ldots, i_{p-1}$ have to lie in V_0, contradicting the assumption.

Since by part a) all connected components are a complete, there exists an edge connection i_0 and a and hence the transposition $(i_0 \; a)$ is contained in H. Now, since

$$(i_\ell \; a) = \sigma^\ell (i_0 \; a) \sigma^{-\ell} \in H$$

for all $\ell \in \{1, \ldots, p-1\}$, there exist edges connecting the vertices $i_1, \ldots, i_\ell$ to a and hence they are all contained in V_0, which gives us the desired contradiction.

Finally, since σ fixes p vertices in a connected component V_0, we know that $|V_0| \geq p > \frac{k}{2}$. But since all connected components contain the same number of vertices, Γ can only contain one connected component, i.e., Γ is connected. This completes the proof by part b).

4. a) Let $y \in V_0$. Since the graph is connected, there is a path from x_0 to every vertex y. Assume the path goes through the vertices $(x_0, x_1, x_2, \ldots, y)$. Since Γ is bipartite, one extremity of every edge lies in V_0 and the other one in V_1. Hence, since $x_0 \in V_0$, we know that $x_1 \in V_1$, $x_2 \in V_0$, $x_3 \in V_1$ and so on. Since y lies in V_0, the path must therefore be of even length. Conversely, by the same argument we see that if there is a path of even length joining x_0 to y, then y has to lie in V_0.

b) Consider a partition $V = V_0 \cup V_1$ of a connected bipartite graph V. If $|V| \leq 1$ we are done. So assume $|V| \geq 2$. Since the graph is connected there exists at least one edge and hence there exists some vertex $x \in V_0$ (since by the definition of a bipartite graph, every edge has one extremity in V_0 and one in V_1). But by part a), the vertex x determines already the bipartite decomposition, since for every other decomposition $V = W_0 \cup W_1$ with $x \in W_0$, we have

$$V_0 = \{y \in V |\text{ there is a path of even length joining } x \text{ to } y\} = W_0$$

and hence $V_1 = W_1$.

c) In this case we have that $W = V$. To prove this, assume that $|V| \geq 2$. Denote

$$W = \{y \in V |\text{ there is a path of even length joining } x_0 \text{ to } y\}.$$

First note that for every $x \in W$,

$$W = \{y \in V |\text{ there is a path of even length joining } x \text{ to } y\}.$$

To check "$\subset$", let $y \in W$. By definition there exists a path of even length joining x_0 to y. But since x is also in W, there exists also a path of even length joining x to x_0. By concatenating these two paths, we see that there is a path of even length joining x to y. For "$\supset$", assume that there exists a path of even length joining x to y. Since $x \in W$, there exists a path of even length joining x_0 to x and hence concatenating this two paths leads to a path of even length joining x_0 to y which shows that $y \in W$.

Denote

$$W' = \{y \in V |\text{ there is a path of odd length joining } x_0 \text{ to } y\}.$$

Obviously, $W' \cup W = V$. We show that, either $W' \cap W = \emptyset$ or $W = W'$. So, assume that $y_0 \in W' \cap W$. Hence there exists a path of odd as well as one of even length joining x_0 to y_0. By combining this two paths, we obtain a path of odd length joining x_0 to itself. Denote this path by γ. Now, for every $y \in W'$ there exists an path of odd length joining x_0 to y. By combining this path with γ we obtain a path of even length joining x_0 to y and hence $y \in W$. For every $y \in W$, there exists a path of even length joining x_0 to y and by combining this path with γ we obtain a path of odd length joining x_0 to y and hence $y \in W'$. Therefore $W = W'$.

Bitte wenden!

Now, note that if $W \cap W' = \emptyset$ there exists no edge with both extremities in W as otherwise there would exist a path of odd length (length 1) joining two points x_1 and x_2 in W and hence a path of odd length joining x_0 to x_2, which means that $x_2 \in W \cap W'$. Analogously, there is no edge with both extremities y_1 and y_2 in W' as otherwise, there exists a path of even length joining x_0 to y_2 which would result in $y_2 \in W \cap W'$. Hence every edge must have one extremity in W and one W'. Since the graph is connected, there needs to exist at least one edge and hence at least one vertex in W and W'. Hence, $W \cap W' = \emptyset$ implies that the graph is bipartite. By contraposition, if the graph is not bipartite, $W \cap W' \neq \emptyset$ and hence $W = W' = V$ (since $W \cup W' = V$).

d) Let Γ be a forest. It is enough to show that every connected component of Γ is bipartite. Hence, assume that Γ is a tree. Assume by contradiction that Γ is not bipartite. Hence $|V| \geq 2$. Then, by part c), $W = W' = V$, where W and W' are defined as in part c). Let $y_1, y_2 \in V$, $y_1 \neq y_2$. Since there exists paths of odd as well as of even length joining y_1 and y_2, these paths can not coincide. One can now see, that thus there must exist a cycle in Γ (the details are left to the reader). Hence Γ can not be a tree, leading to a contradiction.

e) As an exercise for the reader, show that if there are two paths from x to y of length m and n respectively which do not have identical edge sets, i.e., one path contains at least an edge which is not contained in the other one, then there exists a cycle of length $\leq m + n$.

Let Γ be a finite graph. We first show that if Γ is not bipartite, then its girth is finite. We do this by proving the contraposition. So, assume that Γ has infinite girth. Then, by definition, Γ is a forest and hence is bipartite by part d), which completes the proof.

Next, we show that $g := \text{girth}(\Gamma) \leq 2\text{diam}(\Gamma) + 1$. So let c be a cycle in Γ of length g. Let x be a vertex on this cycle and assume by contradiction that there exists another vertex y on C which has distance $\text{diam}(\Gamma) + 1$ from x on the cycle (note that the distance is measured on the cycle C and not on the graph Γ), i.e., $g > 2\text{diam}(\Gamma) + 1$. But then there exists a path in Γ of length $\leq \text{diam}(\Gamma)$ in Γ joining x and y. Furthermore, this path can not have the same edge set as the path γ of length $\text{diam}(\Gamma) + 1$ on the cycle connecting x to y, since γ contains $\text{diam}(\Gamma) + 1$ different edges. Hence there exists a cycle of length $\leq 2\text{diam}(\Gamma) + 1$ by the above exercise, leading to a contradiction.

To see that this bound is best possible, consider the cycle C_m for some odd m. C_m has diameter $\frac{m-1}{2}$ and girth m, hence the equality is sharp.