Solution 3

1. a) This follows directly from the definition of the norm

$$\|M\| = \sup_{\varphi \neq 0} \frac{|\langle M\varphi, \varphi \rangle|}{\|\varphi\|^2}$$

and the fact that M is diagonalizable.

b) Let φ be an eigenvector of M. Denote $m = \max_{x\in\Gamma}|\varphi(x)|$. By the definition of M (averaging), $(M\varphi)(x) \leq m$ for all $x \in \Gamma$. Hence the absolute value of all eigenvectors of M needs to be ≤ 1.

2. a) Assume that φ is the characteristic function of a connected component of Γ, i.e., let $A \subset V$ be a connected component of Γ and let

$$\varphi(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{otherwise.} \end{cases}$$

First consider the case $x \in A$. Then by the definition of the Markov averaging operator M, we have

$$(M\varphi)(x) = \frac{1}{\text{val}(x)} \sum_{\substack{\alpha \in E \\ \text{ep}(\alpha)=\{x,y\}}} \varphi(y) = \frac{1}{\text{val}(x)} \sum_{\substack{\alpha \in E \\ \text{ep}(\alpha)=\{x,y\}}} 1 = 1.$$

Now, we consider the case $x \notin A$. Hence

$$(M\varphi)(x) = \frac{1}{\text{val}(x)} \sum_{\substack{\alpha \in E \\ \text{ep}(\alpha)=\{x,y\}}} \varphi(y) = \frac{1}{\text{val}(x)} \sum_{\substack{\alpha \in E \\ \text{ep}(\alpha)=\{x,y\}}} 0 = 0.$$

Therefore, we can conclude that $M\varphi = \varphi$.

b) Assume that every real-valued element of $\ker(M-1)$ is constant on each connected component of Γ. We want to show that the characteristic functions of the connected components span $\ker(M-1)$.

Hence, let $\psi \in \ker(M-1)$. Then, by the definition of the kernel, $(M-1)\psi = \mathbf{0}$, where $\mathbf{0}$ denotes the zero-function. Since $M-1$ is a real operator, we have that

$$\mathbf{0} = \text{Re}\big((M-1)\psi\big) = (M-1)\text{Re}(\psi)$$

as well as

$$\mathbf{0} = \mathrm{Im}\big((M-1)\psi\big) = (M-1)\mathrm{Im}(\psi).$$

Thus $\mathrm{Re}(\psi)$ and $\mathrm{Im}(\psi)$ are real valued elements of $\ker(M-1)$. By the assumption, they are constant on each connected component and hence they can be written as a linear combination of characteristic functions of connected components of Γ. But then also

$$\psi = \mathrm{Re}(\psi) + i\mathrm{Im}(\psi)$$

can be written as a linear combination of such characteristic functions, which is what we wanted to show.

c) Let φ be a real-valued element of $\ker(M-1)$ and $x_0 \in W$ such that $\varphi(x_0) = m$ where m is the maximum value of φ on the connected component W. Then, by the definition of M,

$$\begin{aligned} 0 = \big((M-1)\varphi\big)(x_0) &= \frac{1}{\mathrm{val}(x_0)} \sum_{\substack{\alpha\in E\\ \mathrm{ep}(\alpha)=\{x_0,x\}}} \varphi(x) - \varphi(x_0) \\ &= \frac{1}{\mathrm{val}(x_0)} \sum_{\substack{\alpha\in E\\ \mathrm{ep}(\alpha)=\{x_0,x\}}} \varphi(x) - m. \end{aligned}$$

Thus

$$\mathrm{val}(x_0)m = \sum_{\substack{\alpha\in E\\ \mathrm{ep}(\alpha)=\{x_0,x\}}} \varphi(x).$$

However, since m is the maximal value of φ on W, the last equality can only hold if for all x connected to x_0, $\varphi(x) = m$.

d) Let $x \in W$. Since W is a connected component, there exists a path from x_0 to x. Now, by induction over the vertices, we can conclude that for every vertex y on the path we have $\varphi(y) = m$. Hence $\varphi(x) = m$.

Hence every real-valued function in $\ker(M-1)$ is constant on every connected component and hence by part b) is spanned by the characteristic functions of the connected components.

e) For a bipartite connected component $W \subset V$ of Γ, let $W_0 \sqcup W_1 = W$ be a bipartite decomposition of W. We have seen that such a decomposition is in fact unique. Let now $\varepsilon_{W,\pm}$ be the functions which is defined by

$$\varepsilon_{W,\pm}(x) = \begin{cases} -1 & \text{if } x \in W_0, \\ 1 & \text{if } x \in W_1, \\ 0 & \text{otherwise.} \end{cases}$$

We claim that $\ker(M+1)$ is spanned by

$$\{\varepsilon_{W,\pm} \mid W \text{ is a bipartite connected component of } \Gamma\}.$$

The proof goes completely analogously to the one for $\ker(M-1)$ except that in part c), one shows that if $\varphi(x_0) = m$, then $\varphi(x) = -m$ for all x connected to x_0 and for part b) and d), one has to assume that φ is of the form

$$\varphi(x) = \begin{cases} -m & \text{if } x \in W_0, \\ m & \text{if } x \in W_1. \end{cases}$$

on any bipartite connected component W of Γ with bipartite decomposition $W_0 \sqcup W_1 = W$ and zero on any non-bipartite connected component.

3. Let $I_k = \{k, k+1, \dots, n\}$ and $I = I_1 \times I_2 \times \dots \times I_{n-1}$. Note that $|I| = n!$ and that $(J_1, \dots, J_{n-1})$ is uniformly distributed in I. Since $|\mathcal{S}_n| = n!$ and since the map

$$I \longmapsto \mathcal{S}_n, \quad (J_1, \dots, J_{n-1}) \longmapsto \sigma_{n-1}$$

is injective and hence a bijection, it is clear that σ_{n-1} is uniformly distributed in $\mathcal{S}_n$.

4. First note that a random walk $(x_n)_{n\in\mathbb{N}}$ is recurrent if and only if for every $x \in \Gamma$

$$\mathbb{E}\big[|\{n > 0 : x_n = x\}|\big] = \sum_{n=0}^{\infty} \mathbb{P}[x_n = x] = \infty.$$

This follows immediately from the definition, as

$$\mathbb{P}\big[|\{n > 0 : x_n = x\}| = \infty\big] = \mathbb{P}[x_n = x \text{ for infinitely many } n] = 1.$$

Exercises a) to c) can then be solved by simple counting arguments. Details are for example carried out in [1].

References

[1] James Norris, Markov Chains, Cambridge Series on Statistical and Probabilistic Mathematics, Cambridge University Press, 1997, Chapter 1.6, available at http://www.statslab.cam.ac.uk/∼james/Markov/s16.pdf