Solution 5

Acknowledgements
The following solutions were kindly provided by Nicolas Müller. The exercise is based on [1], where also further information can be found.

1. a) No solution provided for this exercise.

b) From Linear Algebra, we have the following theorem:

$$\lambda_{n-1} = \max_{\varphi \in L^2_{\mathbb{R}} \setminus \{0\}} \frac{\langle \varphi, (\mathrm{Id} - M)\varphi \rangle}{\langle \varphi, \varphi \rangle}$$

and every maximizing φ is an eigenvector of λ_{n-1}.
We then have

$$\begin{aligned}
2 - \lambda_{n-1} &= 2 - \max_{\varphi \in L^2_{\mathbb{R}} \setminus \{0\}} \frac{\langle \varphi, (\mathrm{Id} - M)\varphi \rangle}{\langle \varphi, \varphi \rangle} \\
&= 2 + \min_{\varphi \in L^2_{\mathbb{R}} \setminus \{0\}} \frac{\langle \varphi, (-\mathrm{Id} + M)\varphi \rangle}{\langle \varphi, \varphi \rangle} \\
&= 2 + \min_{\varphi \in L^2_{\mathbb{R}} \setminus \{0\}} \frac{\langle \varphi, (-2\mathrm{Id} + (\mathrm{Id} + M))\varphi \rangle}{\langle \varphi, \varphi \rangle} \\
&= 2 + \min_{\varphi \in L^2_{\mathbb{R}} \setminus \{0\}} -2 + \frac{\langle \varphi, (\mathrm{Id} + M)\varphi \rangle}{\langle \varphi, \varphi \rangle} \\
&= \min_{\varphi \in L^2_{\mathbb{R}} \setminus \{0\}} \frac{\sum_{\substack{e \in E \text{ with} \\ \mathrm{ep}(e) = \{v,w\}}} (\varphi(v) + \varphi(w))^2}{\sum_{v \in V} \mathrm{val}(v)\varphi(v)^2}
\end{aligned}$$

and if a vector v is a minimizer, then it is also a maximizer in the theorem quoted above and hence an eigenvector of λ_{n-1}.

c) We have

$$\begin{aligned}
2 - \lambda_{n-1} = \min_{x \in \mathbb{R}^V \setminus \{0\}} \frac{\sum_{\{u,v\} \in E} (x_u + x_v)^2}{\sum_{v \in V} \mathrm{val}(x) x_v^2} &\leq \min_{x \in \{-1,0,1\}^V \setminus \{0\}} \frac{\sum_{\{u,v\} \in E} (x_u + x_v)^2}{\sum_{v \in V} \mathrm{val}(x) x_v^2} \\
&\leq 2 \min_{x \in \{-1,0,1\}^V \setminus \{0\}} \frac{\sum_{\{u,v\} \in E} |x_u + x_v|}{\sum_{v \in V} \mathrm{val}(x) |x_v|} = 2\beta(\Gamma)
\end{aligned}$$

where the last inequality follows because for $x, y \in \{-1, 0, 1\}$, we have $x^2 = |x|$ and $|x + y| \in \{0, 1, 2\}$ and therefore $(x + y)^2 \leq 2|x + y|$.

d) In general we have

$$|y_v^t + y_u^t| \in \{0, 1, 2\}$$

and therefore

$$\mathbb{E}[|y_v^t + y_u^t|] = 1 \cdot P(|y_v^t + y_u^t| = 1) + 2 \cdot P(|y_v^t + y_u^t| = 2)$$

Suppose that $x_v \cdot x_u \geq 0$. Then

$$|y_u^t + y_v^t| = \begin{cases} 2 & |t| \leq \min(|x_u|, |x_v|) \\ 1 & \min(|x_u|, |x_v|) < |t| \leq \max(|x_u|, |x_v|) \\ 0 & |t| > \max(|x_u|, |x_v|) \end{cases}$$
$$= \begin{cases} 2 & t^2 \leq \min(x_u^2, x_v^2) \\ 1 & \min(x_u^2, x_v^2) < t^2 \leq \max(x_u^2, x_v^2) \\ 0 & t^2 > \max(x_u^2, x_v^2) \end{cases}$$

Since t^2 is uniformly distributed in $]0, 1]$, it follows that

$$P(|y_v^t + y_u^t| = 2) = \min(x_u^2, x_v^2)$$
$$P(|y_v^t + y_u^t| = 1) = \max(x_u^2, x_v^2) - \min(x_u^2, x_v^2)$$

And therefore

$$\mathbb{E}[|y_u^t + y_v^t|] = \max(x_u^2, x_v^2) + \min(x_u^2, x_v^2) = x_u^2 + x_v^2$$
$$= |x_u|^2 + |x_v|^2 \leq (|x_u| + |x_v|)(|x_u| + |x_v|) \overset{\text{same sign}}{=} |x_u + x_v|(|x_u| + |x_v|)$$

Suppose now that $x_v \cdot x_u \leq 0$. Then

$$|y_u^t + y_v^t| = \begin{cases} 1 & \min(x_u^2, x_v^2) < t^2 \leq \max(x_u^2, x_v^2) \\ 0 & \max(x_u^2, x_v^2) < t^2 \vee \min(x_u^2, x_v^2) \geq t^2 \end{cases}$$

and we get

$$\mathbb{E}[|y_u^t + y_v^t|] = \max(x_u^2, x_v^2) - \min(x_u^2, x_v^2) = |x_u^2 - x_v^2| = |x_u - x_v||x_u + x_v| \leq |x_u + x_v|(|x_u| + |x_v|)$$

In conclusion, for arbitrary signs of x_u and x_v we get

$$\mathbb{E}[|y_u^t + y_v^t|] \leq |x_u + x_v|(|x_u| + |x_v|)$$

e) We have

$$\mathbb{E}\left[\sum_{\{u,v\} \in E} |y_u^t + y_v^t|\right] = \sum_{\{u,v\} \in E} \mathbb{E}[|y_u^t + y_v^t|] \leq \sum_{\{u,v\} \in E} |x_u + x_v|(|x_u| + |x_v|)$$
$$\underset{\text{Cauchy-Schwarz in } \mathbb{R}^n}{=} \sqrt{\sum_{\{u,v\} \in E} \underbrace{|x_u + x_v|^2}_{=(x_u + x_v)^2}} \sqrt{\sum_{\{u,v\} \in E} (|x_u| + |x_v|)^2}.$$

Siehe nächstes Blatt!

Additionally,

$$\mathbb{E}\left[\sum_{v\in V}\text{val}(v)|y_v^t|\right] = \sum_{v\in V}\text{val}(v)\mathbb{E}[|y_v^t|] = \sum_{v\in V}\text{val}(v)P(x_v^2 \geq t^2)$$
$$= \sum_{v\in V}\text{val}(v)x_v^2.$$

Also

$$\sum_{\{u,v\}\in E}(|x_u| + |x_v|)^2 \overset{\text{Young's inequality}}{\leq} 2\sum_{\{u,v\}\in E}x_u^2 + x_v^2 = 2\sum_{v\in V}\text{val}(v)x_v^2.$$

For the final part we have

$$\frac{\mathbb{E}\left[\sum_{\{u,v\}\in E}|y_u^t + y_v^t|\right]}{\mathbb{E}\left[\sum_{v\in V}\text{val}(v)|y_v^t|\right]} \leq \frac{\sqrt{\sum_{\{u,v\}\in E}(x_u + x_v)^2}\sqrt{\sum_{\{u,v\}\in E}(|x_u| + |x_v|)^2}}{\sqrt{\sum_{v\in V}\text{val}(v)x_v^2}^2}$$
$$\leq \frac{\sqrt{2}\cdot\sqrt{\sum_{\{u,v\}\in E}(x_u + x_v)^2}}{\sqrt{\sum_{v\in V}\text{val}(v)x_v^2}}.$$

f) We have

$$P\left(\frac{X}{Y} \leq \frac{\mathbb{E}[X]}{\mathbb{E}[Y]}\right) = P\left(X\mathbb{E}[Y] \leq \mathbb{E}[X]Y\right) = P(Y\mathbb{E}[X] - X\mathbb{E}[Y] \geq 0)$$

and

$$\mathbb{E}[\mathbb{E}[X]Y - \mathbb{E}[Y]X] = \mathbb{E}[X]\mathbb{E}[Y] - \mathbb{E}[Y]\mathbb{E}[X] = 0$$

It follows that $P(\mathbb{E}[X]Y - \mathbb{E}[Y]X \geq 0) > 0$ which finishes the proof.

g) Let x be a minimizer of

$$\min_{x\in\mathbb{R}^n\setminus\{0\}} \frac{\sum_{\substack{e\in E \text{ with} \\ \text{ep}(e)=\{v,w\}}}(x_v + x_w)^2}{\sum_{v\in V}\text{val}(v)x_v^2}.$$

Since $x \neq 0$ and the above expression is obviously invariant under multiplying x with a nonzero scalar, we can choose x such that $\|x\|_\infty = 1$. Define

$$X = \sum_{\{u,v\}\in E}|y_u^t + y_v^t|,$$
$$Y = \sum_{v\in V}\text{val}(v)|y_v^t|,$$

Bitte wenden!

where y and t are as defined in d). Because $\|x\|_\infty = 1$, it is impossible that all $\forall v \in V : y_v^t = 0$. Therefore $P(Y > 0) = 1$. By f), we have that

$$P\left(\frac{X}{Y} \leq \frac{\mathbb{E}[X]}{\mathbb{E}[Y]}\right) > 0.$$

Hence, there exists a $t_0 \in]0, 1]$ such that $X(t_0)/Y(t_0) \leq \mathbb{E}[X]/\mathbb{E}[Y]$. From the definitions of β and y^{t_0} we get that

$$\beta(\Gamma) \leq \frac{X(t_0)}{Y(t_0)} \leq \frac{\mathbb{E}[X]}{\mathbb{E}[Y]} \stackrel{\text{e)}}{\leq} \sqrt{2\frac{\sum_{\{u,v\}\in E}(x_u + x_v)^2}{\sum_{v\in V} \text{val}(x) x_v^2}} = \sqrt{2(2 - \lambda_{n-1})}.$$

References

[1] Luca Trevisan, Lecture Notes “Graph Partitioning, Expanders and Spectral Methods”, Lecture 5, available at http://www.eecs.berkeley.edu/∼luca/expanders2016/index.html