Solution 7

1. a) The proof goes analogously to the proof that $G = \mathbb{R}$ does not have property (T) as seen in the lecture.

2. First the "if" part: Let A be a finitely generated abelian group and let S be a fixed generating set of G. Consider a unitary representation

$$\varrho\colon A \longrightarrow \mathcal{U}(\mathcal{H})$$

which possesses almost fixed vectors with respect to S. Hence there exists $v \in \mathcal{H}$ with $\|v\| = 1$ and

$$\max_{s\in S}\|\varrho(s)v - v\| < \sqrt{2}. \tag{1}$$

For $s \in S$, we write $\varrho(s)v = \lambda_s v + w_s$ where λ_s is a complex number and w_s is a vector perpendicular to v. Since ϱ is unitary we have that

$$\begin{aligned}
\|\varrho(s)v - v\|^2 &= \langle \varrho(s)v - v, \varrho(s)v - v\rangle \\
&= \langle \varrho(s)v, \varrho(s)v\rangle - \langle \varrho(s)v, v\rangle - \langle v, \varrho(s)v\rangle + \langle v, v\rangle \\
&= \langle v, v\rangle - \langle \lambda_s v + w_s, v\rangle - \langle v, \lambda_s v + w_s\rangle + \langle v, v\rangle \\
&= 2\langle v, v\rangle - \langle \lambda_s v, v\rangle - \langle v, \lambda_s v\rangle \\
&= 2\langle v, v\rangle - (\overline{\lambda_s} + \lambda_s)\langle v, v\rangle \\
&= 2\|v\|^2 - 2\mathrm{Re}(\lambda_s)\|v\|^2 \\
&= 2(1 - \mathrm{Re}\lambda_s).
\end{aligned}$$

Hence, by (1), we get that $\mathrm{Re}\lambda_s > 0$. Define

$$u = \sum_{g\in G} \varrho(g)v.$$

We compute

$$\begin{aligned}
\langle v, u\rangle = \langle v, \sum_{g\in G}\varrho(g)v\rangle &= \sum_{g\in G}\langle v, \varrho(g)v\rangle = \sum_{g\in G}\langle v, \lambda_g v + w_g\rangle \\
= \sum_{g\in G}\lambda_g\langle v, v\rangle &= \sum_{g\in G}\lambda_g
\end{aligned}$$

and therefore

$$\mathrm{Re}(\langle v, u\rangle) = \sum_{g\in G} \mathrm{Re}\lambda_g > 0,$$

which implies $u \neq 0$. Also, for $g' \in G$

$$\varrho(g')u = \sum_{g\in G} \varrho(g')\varrho(g)v = \sum_{g\in G} \varrho(g' \circ g)v = \sum_{g\in G} \varrho(g)v = u.$$

Therefore, u is a fixed vector.

Now, we prove the "only if" part: We prove the contraposition, i.e., if A is infinite then A does not have property (T). So let A be infinite and assume by contradiction that A has property (T). By the classification of finitely generated abelian groups, A is of the form

$$\mathbb{Z}^n \oplus \mathbb{Z}/q_1\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/q_t\mathbb{Z}$$

where $q_1, \ldots, q_t$ are prime powers, $t \geq 0$ and $n \geq 1$ (since A is infinite). Hence there exists a surjective homomorphism $A \longrightarrow \mathbb{Z}$ and hence by exercise 1 b), $\mathbb{Z}$ possesses property (T) which is a contradiction to exercise 1 a).

3. Let $\pi\colon G \longrightarrow G/H$ be the projection map and assume that G possesses property (T). Let

$$\varrho\colon G/H \longrightarrow \mathcal{U}(\mathcal{H})$$

be a unitary representation of G/H and assume that ϱ possesses an almost fixed vector. Then

$$\varrho \circ \pi\colon G \longrightarrow \mathcal{U}(\mathcal{H})$$

possesses also an almost fixed vector. Since G possesses property (T), there exists an fixed vector $v \in \mathcal{H}$ of $\varrho \circ \pi$, i.e. for all $g \in G$, $\varrho \circ \pi(g)(v) = v$. But since π is surjective, there exists for every $h \in G/H$ a $g \in G$ such that $\pi(g) = h$. Hence $\varrho(h)(v) = \varrho \circ \pi(g)(v) = v$. Hence v is also a fixed vector for ϱ. This shows that G/H possesses property (T).

4. Let F_k be the free group on k generators. Assume by contradiction that F_k has property (T). Since the commutator subgroup $[F_k, F_k]$ is a normal subgroup of F_k, we know by by exercise 3, that also $F_k/[F_k, F_k]$ has property (T). Because $F_k/[F_k, F_k]$ is a finitely generated abelian group with property (T), we know by exercise 2 that it needs to be finite, which is obviously not the case. Hence we get a contradiction.

5. No solution provided.

Siehe nächstes Blatt!

6. See Proposition B.1.3 of the lecture notes "Expander Graphs" by E. Kowalski.

7. By exercise 6, Γ_3 is a free group on 2 generators and hence by exercise 4, Γ_3 does not have property (T). But then, by exercise 5, $\mathrm{SL}_2(\mathbb{Z})$ also does not have property (T).