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Chap. IV Elliptic regularity

$\Omega \subset \mathbb{R}^n$ open, bounded.

$$\mathcal{L}u = \sum_{|\sigma| \le m} a_\sigma \partial^\sigma u$$

$a_\sigma : \overline{\Omega} \to \mathbb{R}$ smooth for $\sigma \in \mathbb{N}_0^n$, $|\sigma| \le m$.

Def'n: The formal adjoint operator of $\mathcal{L}$ is the operator

$$\mathcal{L}^* v := \sum_{|\sigma| \le m} (-1)^{|\sigma|} \partial^\sigma (a_\sigma v)$$

Exercise: $\forall \varphi, \psi \in C_0^\infty(\Omega)$

$$\int_\Omega \psi \mathcal{L}^* \varphi = \int_\Omega \varphi \cdot \mathcal{L}\psi.$$

Def'n: $f \in \mathcal{L}^p(\Omega)$

A fct'n $u \in \mathcal{L}^p(\Omega)$ is called a weak solution of the eq'n:

$$(1) \quad \mathcal{L}u = f$$

if

$$(2) \quad \int_\Omega u \mathcal{L}^* \varphi = \int_\Omega f \varphi.$$

Lemma 1: $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$.

$\mathcal{L}, \mathcal{L}^*$ as above

Assume $\exists c > 0 \ \forall \varphi \in C_0^\infty(\Omega)$

$$\|\varphi\|_{\mathcal{L}^q} \le c \|\mathcal{L}^* \varphi\|_{\mathcal{L}^q}$$

$\Rightarrow \forall f \in \mathcal{L}^p(\Omega) \ \exists u \in \mathcal{L}^p(\Omega)$ weak sol'n of (1).

Pf: $\mathcal{L}^p(\Omega) \cong (\mathcal{L}^q(\Omega))^*$.

$$\mathcal{L}^q(\Omega) \supset \{\mathcal{L}^* \varphi \mid \varphi \in C_0^\infty(\Omega)\} =: V.$$

Define $\Lambda : Y \to \mathbb{R}$ by

$$\Lambda(\mathcal{L}^* \varphi) := \int_\Omega f \varphi \quad \text{for } \varphi \in C_0^\infty(\Omega)$$

well-defined and bounded.

$$|\Lambda(\mathcal{L}^* \varphi)| = \left| \int_\Omega f \varphi \right| \le \|f\|_{\mathcal{L}^p} \|\varphi\|_{\mathcal{L}^q}$$
$$\le C \|f\|_{\mathcal{L}^p} \|\mathcal{L}^* \varphi\|_{\mathcal{L}^q}$$

$\overset{H\text{-}B}{\Longrightarrow}$ Λ extends to a bnded lin. functional

$$\Lambda : \mathcal{L}^q(\Omega) \to \mathbb{R}$$

$\Rightarrow$ $\exists!\, u \in \mathcal{L}^p(\Omega)$ $\forall v \in \mathcal{L}^q(\Omega)$: $\Lambda(v) = \int_\Omega u v$.

$\overset{v = \mathcal{L}^*\varphi}{\Longrightarrow}$ $\int_\Omega u \mathcal{L}^* \varphi = \Lambda(\mathcal{L}^* \varphi) = \int_\Omega f \varphi$

qed.

Constant coeff. : $a_\sigma \in \mathbb{C}$, $|\sigma| = m$.

$$\mathcal{L} = \sum_{|\sigma| = m} a_\sigma \partial^\sigma$$

<u>Def'n</u>: $\mathcal{L}$ is called <u>elliptic</u> if the symbol $\sum_{|\sigma|=m} a_\sigma \xi^\sigma \neq 0 \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}$.

<u>Examples:</u>

Laplace $\quad \Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ elliptic.

Cauchy-Riemann $\quad \mathcal{L} = \frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2}$ on $\mathbb{C}^2$ elliptic

heat eq $\quad \mathcal{L} = \frac{\partial}{\partial t_2} - \Delta$ on $\mathbb{R}^{n+1}$

wave eq $\quad \mathcal{L} = \frac{\partial^2}{\partial t^2} - \Delta$

Schrödinger $\quad \mathcal{L} = \frac{\partial}{\partial t} - i\Delta$

$\Bigg\}$ not elliptic.

②

Lemma 2 $1 < p < \infty$.
$\mathcal{L}$ homogeneous elliptic op. on $\mathbb{R}^n$ of order m with constant coeff.
$\Rightarrow \exists\, c > 0 \quad \forall\, v \in \mathcal{S}(\mathbb{R}^n, \mathbb{C})$
$$\| \partial^m v \|_{\mathcal{L}^p} \leq c \, \| \mathcal{L} v \|_{\mathcal{L}^p}.$$

__Proof__: $v \in \mathcal{S}(\mathbb{R}^n, \mathbb{C})$, $\beta \in \mathbb{N}^n$, $|\beta| = m$.
$$\widehat{\partial^\beta v}(\xi) = (i\xi)^\beta \hat{v}(\xi)$$
$$\widehat{\mathcal{L}v}(\xi) = \sum_{|\sigma|=m} a_\sigma \widehat{\partial^\sigma v}(\xi)$$
$$= \sum_{|\sigma|=m} a_\sigma (i\xi)^\sigma \hat{v}(\xi)$$
$$\Rightarrow \widehat{\partial^\beta v}(\xi) = \underbrace{\frac{\xi^\beta}{\sum_\sigma a_\sigma \xi^\sigma}}_{m_\beta(\xi)} \widehat{\mathcal{L}v}(\xi)$$

$m_\beta : \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$.

If $\delta := \min_{\substack{\xi \in \mathbb{R}^n \\ |\xi| = 1}} \left| \sum_{|\sigma|=m} a_\sigma \xi^\sigma \right| > 0$ ellipt.
$$\Rightarrow \left| \sum_{|\sigma|=m} a_\sigma \xi^\sigma \right| \geq \delta |\xi|^m.$$
$$\Rightarrow |m_\beta(\xi)| \leq \frac{1}{\delta} \quad \forall \xi.$$

$\forall\, \alpha \in \mathbb{N}_0^n \ \exists\, C_\alpha > 0 \ \forall\, \xi \in \mathbb{R}^n \setminus \{0\}$
$$|\partial^\alpha m_\beta(\xi)| \leq \frac{C_\alpha}{|\xi|^{|\alpha|}}.$$

$\overset{\text{multiplier thm.}}{\Longrightarrow} \exists\, c > 0 \quad \forall\, f \in \mathcal{L}^p(\mathbb{R}^n, \mathbb{C}) \cap \mathcal{L}^2(\mathbb{R}^n, \mathbb{C})$
$$\| T_{m_\beta} f \|_{\mathcal{L}^p} \leq c \, \| f \|_{\mathcal{L}^p}$$

$$\widehat{T_{m_\beta} f} = m_\beta \cdot \hat{f}$$

$$T_{m_\beta}(\mathcal{L} v) = \partial^\beta v \quad \Rightarrow \text{ Lemma 2}$$

qed.

Return to $a_\sigma \in \mathbb{R}$, $|\sigma| = m$

$$\mathcal{L} = \sum_{|\sigma|=m} a_\sigma \partial^\sigma \quad \text{elliptic. on } \Omega \subset \mathbb{R}^n \text{ op, bnded}$$

$$\mathcal{L}^* = (-1)^m \mathcal{L}$$

$$\overset{\text{Lemma 2}}{\Longrightarrow} \quad \| \partial^m v \|_{\mathcal{L}^p(\Omega)} \leq C \| \mathcal{L}^* v \|_{\mathcal{L}^p} \quad \forall v \in C_0^\infty(\mathbb{R}^n)$$

Poincaré:

$$\| v \|_{\mathcal{L}^p} \leq C_1 \| \partial^m v \|_{\mathcal{L}^p}$$
$$\leq C_1 C_2 \| \mathcal{L}^* v \|_{\mathcal{L}^p}$$

$\Rightarrow$ Lemma 1 applies. $\Rightarrow$ weak sol'ns exist.

<u>Def'n</u> $\Omega \subset \mathbb{R}^n$ bnded, open

A diff. operator $\mathcal{L}$ on Ω has <u>divergence structure</u> if it can be written as

$$\mathcal{L} v = \sum_{|\sigma|,|\rho| \leq m} (-1)^{|\sigma|} \partial^\sigma (a_{\sigma\rho} \partial^\rho v)$$

where $a_{\sigma\rho} = a_{\rho\sigma} \in C^\infty(\overline{\Omega}, \mathbb{R})$

<u>Example</u>: $m = 1$

$$\mathcal{L} v = \sum_{i,j=1}^n \partial^i (a_{ij} \partial^j v) + c v.$$

$$a_{ij} = a_{ji} \in C^\infty(\overline{\Omega}), \; c \in C^\infty(\overline{\Omega}).$$

<u>Rmk</u>: $\mathcal{L}$ div. structure $\Rightarrow$ $\mathcal{L}^* = \mathcal{L}$.

<u>Def'n</u>: $\mathcal{L}$ in divergence form is called <u>elliptic</u> if

$$\sum_{|\sigma|=|\rho|=m} a_{\sigma\rho}(x) \xi^{\sigma+\rho} \neq 0 \quad \forall x \in \Omega, \; \forall \xi \in \mathbb{R}^n \setminus \{0\}.$$

③

<u>strongly elliptic:</u>

if $\exists \delta > 0 \ \forall x \in \Omega \ \forall \xi \in \mathbb{R}^n$

$$\sum_{|\sigma|=|\rho|=m} a_{\sigma\rho} \xi^{\rho+\sigma} \geq \delta |\xi|^{2m}$$

2^{nd} order ell. op.'s of the form:

$$(1) \quad \boxed{\mathcal{L}u = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial u}{\partial x_j} \right)}$$

$\Omega \subset \mathbb{R}^n$ bdded, open, $a_{ij} = a_{ji} \in C^1(\bar{\Omega})$

$$(2) \quad \boxed{\sum a_{ij}(x) \xi_i \xi_j \geq \delta |\xi|^2 \quad \forall x \in \Omega \ \forall \xi \in \mathbb{R}^n}$$

<u>Dirichlet Problem</u>

Given $f : \Omega \to \mathbb{R}$ find a. solution $u : \bar{\Omega} \to \mathbb{R}$ of

$$(3) \begin{cases} \mathcal{L}u = f \\ u|_{\partial\Omega} = 0 \end{cases}$$

[Given $f \in \mathcal{L}^p(\Omega)$ find $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ s.t. $\mathcal{L}u = f$

<u>Thm 1</u> $\Omega \subset \mathbb{R}^n$ bdded open domain, C^∞-bndry $a_{ij} = a_{ji} \in C^1(\bar{\Omega})$ satisfying (2)

$\Rightarrow$ the operator $\mathcal{L} : W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \to \mathcal{L}^p(\Omega)$ given by (1) is bijective i.e. $\forall f \in \mathcal{L}^p(\Omega) \ \exists! u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ s.t. $\mathcal{L}u = f$.

Moreover $\exists c > 0 \ \forall u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$

$$\|u\|_{W^{2,p}(\Omega)} \leq c \|\mathcal{L}u\|_{\mathcal{L}^p(\Omega)}.$$

<u>Def'n</u>: A fct'n $u \in W_0^{1,p}(\Omega)$ is called a <u>weak sol'n</u> of (3) if:

$$(4) \quad \boxed{\sum_{i,j=1}^{n} \int_\Omega \frac{\partial \varphi}{\partial x_i} a_{ij} \frac{\partial u}{\partial x_j} = \int_\Omega f \varphi} \quad \forall \varphi \in C_0^\infty(\Omega)$$

<u>Strategy</u>:

1) Find a weak sol'n $u \in W_0^{1,p}(\Omega)$ of (3)
2) Then prove $u \in W^{2,p}(\Omega)$.

<u>Rmk</u>: 1) a) If $u \in W_0^{1,p}(\Omega)$, $v \in W^{1,p}(\Omega)$

$$\int \left((\partial_i u) v + u (\partial_i v) \right) = 0.$$

b) If $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, $\mathcal{L}u = f$.

then $\sum_{i,j} a_{ij} \frac{\partial u}{\partial x_j} \in W^{1,p}(\Omega)$ so

$$\int_\Omega \varphi f = \int_\Omega \varphi \mathcal{L} u = \int_\Omega \sum_{i,j=1}^{n} a_{ij} \frac{\partial \varphi}{\partial x_j} \frac{\partial u}{\partial x_i}$$

Hence u is a weak sol'n of (3).

2) The formula

$$B(\varphi, \psi) = \sum_{i,j=1}^{n} \int_\Omega \frac{\partial \varphi}{\partial x_i} a_{ij} \frac{\partial \psi}{\partial x_j}$$

defines a bilinear form.

$$B : W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega) \to \mathbb{R} \qquad \frac{1}{p} + \frac{1}{q} = 1.$$

$$|B(\varphi, \psi)| \leq c \, \|\nabla \varphi\|_{L^p} \, \|\nabla \psi\|_{L^q}$$

3) Poincaré inequality

$$\|\nabla u\|_{L^p} \leq \|u\|_{W^{1,p}} \leq (1 + \mathrm{diam}(\Omega)) \|\nabla u\|_{L^p}.$$

So the fct'n:

$$W_0^{1,p}(\Omega) \to \mathbb{R} \; : u \mapsto \|\nabla u\|_{L^p}$$

is a norm on $W_0^{1,p}(\Omega)$ equivalent to the std. norm.

④

4) Denote by

$$W^{-1,p}(\Omega) = \left(W_0^{1,q}(\Omega)\right)^*, \quad \frac{1}{p}+\frac{1}{q}=1$$

$$\Lambda \in W^{-1,p}(\Omega)$$

$\Lambda : W_0^{1,q} \to \mathbb{R}$ bounded lin. fctnl.

$$\|\Lambda\|_{W^{-1,p}} = \sup_{\substack{\varphi \not\equiv 0 \\ \varphi \in C_0^\infty(\Omega)}} \frac{\Lambda(\varphi)}{\|\nabla\varphi\|_{L^q}}$$

5) Every $v \in W_0^{1,p}(\Omega)$ determines a bounded lin. fct'nal

$$\Lambda_v : W_0^{1,q}(\Omega) \to \mathbb{R}$$

$$\Lambda_v(\varphi) = B.(v,\varphi) \quad \text{for } \varphi \in W_0^{1,q}(\Omega).$$

So this defines a bounded lin. op

$$\begin{cases} W_0^{1,p}(\Omega) \to W^{-1,p}(\Omega) \\ v \mapsto \Lambda_v \end{cases}$$

$$\begin{array}{ccc} W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) & \xrightarrow{L} & L^p(\Omega) \\ \downarrow & (\hookrightarrow \text{Rmk 1}). & \downarrow \leftarrow \text{inj. comp, dense image} \\ W_0^{1,p}(\Omega) & \xrightarrow{L} & W^{-1,p}(\Omega) \end{array}$$

$f \mapsto \Phi_f$ s.t. $\Phi_f(v) = \int_\Omega f \cdot v$

Thm 2: The op. $L: W_0^{1,p}(\Omega) \to W^{-1,p}(\Omega)$ is bijective.
Moreover, $\exists\, c > 0 \quad \forall v \in W_0^{1,p}(\Omega)$:

$$\|v\|_{W_0^{1,p}(\Omega)} \le c\,\|Lv\|_{W^{-1,p}(\Omega)} = c \sup_{0 \ne \varphi \in C_0^\infty(\Omega)} \frac{|B(v,\varphi)|}{\|\nabla\varphi\|_{L^q}}.$$

Pf of Thm 2 for p=2:

$B: W_0^{1,2} \times W_0^{1,2} \to \mathbb{R}$ bilinear, symmetric

$$C \|\nabla u\|_{L^2}^2 \geq B(u,u) = \sum_{i,j} \int_\Omega \frac{\partial u}{\partial x_i} a_{ij} \frac{\partial u}{\partial x_j}$$

$$\geq \delta \|\nabla u\|_{L^2}^2 .$$

Lax-Milgram Thm.

$\|u\|_B := \sqrt{B(u,u)}$ norm on $W_0^{1,2}(\Omega)$

Given $\Phi : W_0^{1,2} \to \mathbb{R}$

$\overset{\text{Riesz}}{\Longrightarrow}$ $\exists!\, u \in W_0^{1,2}(\Omega)$ $\forall \varphi \in W_0^{1,2}(\Omega)$

$$B(u,\varphi) = \Phi(\varphi).$$

qed.

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Thm 3: $1 < p < \infty$. Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set with smooth boundary. Let $a_{ij} = a_{ji} \in C^1(\bar{\Omega})$ $i,j = 1, \ldots, n$, s.t.

$$\sum_{i,j=1}^n a_{ij}(x)\, \xi_i \xi_j \geq \delta |\xi|^2 \quad \forall x \in \bar{\Omega}, \forall \xi \in \mathbb{R}^n$$

Then $\exists\, c > 0$ s.t. $\forall u \in W_0^{1,p}(\Omega)$ $\forall f_0, f_1, \ldots, f_n \in L^p(\Omega)$ satisfying:

$$(1) \quad \boxed{\int_\Omega \sum_{i,j=1}^n \frac{\partial u}{\partial x_i} a_{ij} \frac{\partial \varphi}{\partial x_j} = \int_\Omega f_0 \varphi + \sum_{i=1}^n \int_\Omega f_i \frac{\partial \varphi}{\partial x_i}}$$

$\forall \varphi \in W_0^{1,q}(\Omega)$, $\frac{1}{p} + \frac{1}{q} = 1$.

We have:

$$(2) \quad \boxed{\|\nabla u\|_{L^p} \leq c \left(\sum_{i=0}^n \|f_i\|_{L^p} + \|u\|_{L^p} \right)}$$

Proof: __Claim:__ $\forall x_0 \in \bar{\Omega}$, $\exists$ open nbhd $U_0 \subset \mathbb{R}^n$ of x_0 $\exists c_0 > 0$ $\forall u \in W_0^{1,p}(\Omega)$ $\forall f_i \in L^p(\Omega)$, we have

(1) and $\operatorname{supp}(u) \subset U_0$, $\operatorname{supp}(f_i) \subset U_0$ $\forall i$

$\Longrightarrow$ (2) with $c = c_0$.

(5)

Claim $\Rightarrow$ Thm 3

Choose a smooth cut-off fct'n $\beta : \mathbb{R}^n \to [0,1]$ with $\operatorname{supp}(\beta) \subset U_0$. Define

$$\begin{cases} \tilde{v} = \beta v \\ \tilde{f}_0 = \beta f_0 + \sum_{i=1}^{n} \partial_i \beta \cdot f_i + v \sum_{i,j} \partial_i (a_{ij} \partial_j \beta) \\ \tilde{f}_i = \beta f_i + 2 v \sum_{j=1}^{n} a_{ij} \partial_j \beta \end{cases}$$

$\overset{\tilde{\varphi} = \beta\varphi}{\Longrightarrow}$ $\tilde{v}, \tilde{f}_i$ satisfy (1)

$\overset{\text{Claim}}{\Longrightarrow}$ $\|\nabla \tilde{v}\|_{L^p} \le c_0 \left(\|\tilde{v}\|_{L^p} + \sum_{i=0}^{n} \|\tilde{f}_i\|_{L^p} \right)$

$$\le c_1 \left(\|v\|_{L^p} + \sum_{i=0}^{n} \|f_i\|_{L^p} \right)$$

Partition of unity:

$\overline{\Omega}_0 \subset \bigcup_{i=1}^{N} U_i$, Claim holds for U_i

Choose $\beta_i \in C_0^\infty(\mathbb{R}^n)$, $\operatorname{supp}(\beta_i) \subset U_i$, $\sum_{i=1}^{N} \beta_i(x) = 1 \ \forall x \in \overline{\Omega}$.

$$\Rightarrow \|\nabla v\|_{L^p} \le \sum_{i=1}^{N} \|\nabla(\beta_i v)\|_{L^p} \le \underbrace{\left(\sum_{i=1}^{N} c_i\right)}_{c} \left(\|v\|_{L^p} + \sum_{i=0}^{n} \|f_i\|_{L^p} \right)$$

Proof of Claim (4 Steps)

$K : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ fund. sol'n of Laplace

$K_i(x) = \frac{\partial K}{\partial x_i}(x) = \frac{x_i}{\omega_i |x|^n}$.

Case 1: $x_0 \in \Omega$ (not in $\partial\Omega$) $\quad a_{ij}(x) = \delta_{ij}$

$$v := \sum_{i=1}^{n} K_i * f_i - K * f_0$$

$$\nabla v = \sum_{i=1}^{n} \nabla(K_i * f_i) - \nabla(K * f_0)$$

By Calderón - Zygmund $\le$, $\exists c_0 > 0 \ \forall f_i$

$$\|\nabla v\| \le c_0 \sum_{i=0}^{n} \|f_i\|_{L^p}.$$

Moreover $\forall \varphi \in C_0^\infty(\Omega)$:

$$\int_\Omega \langle \nabla v, \nabla \varphi \rangle = - \int_\Omega v \, \Delta \varphi$$

$$= - \sum_i \int_\Omega (K_i * f_i) \Delta\varphi + \int_\Omega (K * f_0) \Delta \varphi$$

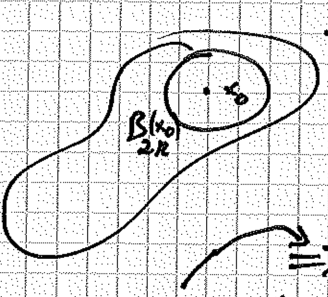
parity of K, K_j $\quad = \sum_i \int_\Omega f_i (K_i * \Delta\varphi) + \int_\Omega f_0 (K * \Delta\varphi)$

By Poisson identities: $K_j * \Delta\varphi = \partial_i (K * \Delta \varphi)$
$\qquad\qquad\qquad\qquad\qquad\qquad\quad = \partial_i \varphi.$

$$\Rightarrow \int_\Omega \langle \nabla v, \nabla \varphi \rangle = \sum_i \int_\Omega f_i \, \partial_i \varphi + \int_\Omega f_0 \varphi$$
$$\overset{(1)}{=} \int \langle \nabla u, \nabla \varphi \rangle .$$

$$\Rightarrow \int_\Omega (v-u)\, \Delta \varphi = 0 \qquad \forall\, \varphi \in C_0^\infty(\Omega)$$

$\overset{\text{Weyl's}}{\underset{\text{lemma}}{\Rightarrow}}$ $v-u$ is harmonic.



$B_{2r}(x_0) \subset \Omega$.

$$\underset{|x-x_0|<r}{\Rightarrow} \quad |\nabla u(x) - \nabla v(x)| \le \frac{n+1}{r} \frac{1}{\mathrm{vol}(B_r)} \int_{B_r(x_0)} |u-v|$$

ineq. in Chapter 1

$$\le \frac{n+1}{r} \left(\frac{1}{\mathrm{vol}(B_r)} \right)^{1/p} \| (u-v) \|_{L^p(B_{2r}(x_0))}$$

$$\Rightarrow \| \nabla u - \nabla v \|_{L^p(B_r(x_0))}$$

$\Omega \subset B_R(x_0)$

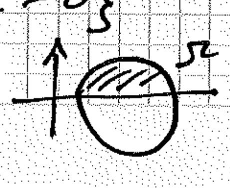
$$\le \frac{n+1}{r} \; \| u - v \|_{L^p(\Omega)}$$

$$\le \frac{n+1}{r} \left(\|u\|_{L^p} + \| K * f_0 \|_{L^p(\Omega)} + \sum_{i=1}^n \| K_i * f_i \|_{L^p(\Omega)} \right)$$

$$\underset{K, K_i \in L^1_{loc}}{\overset{\text{Young}}{\le}} \frac{n+1}{r} \left(\|u\|_{L^p} + \|K\|_{L^1(B_{2R})} \|f_0\|_{L^p} + \sum_{i=1}^n \|K_i\|_{L^1(B_{2R})} \|f_i\|_{L^p} \right)$$

$$\le c_1 \left(\|u\|_{L^p} + \sum_{i=0}^n \|f_i\|_{L^p} \right)$$

<u>Case 2</u> $\quad x_0 \in \partial\Omega$. Assume $x_0 \in \mathbb{R}^{n-1} \times \{0\}$
$\mathcal{U}_0 = \{ x \in \mathbb{R}^n \mid |x - x_0| < \varepsilon \}$
$\Omega \cap \mathcal{U}_0 = \{ x \in \mathcal{U}_0 \mid x_n > 0 \}$
$a_{ij} = \delta_{ij}$



⑥

Abbreviate

$$f := \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \in \mathcal{L}^p(\Omega, \mathbb{R}^n)$$

Define $\quad R = \begin{pmatrix} 1 & \ddots & 0 \\ 0 & 1 & \\ & & -1 \end{pmatrix}$

$$R\,x = (x_1, \dots, x_{n-1}, -x_n)$$

Extend u, f_i to $\mathcal{U}_0$ by:

$$\tilde{u}, \hat{f}_i : \mathcal{U}_0 \to \mathbb{R}$$

$$\tilde{u}(x) = \begin{cases} u(x) & , x_n \geq 0 \\ -u(Rx) & , x_n < 0. \end{cases}$$

$$\hat{f}(x) = \begin{cases} f(x) & , x_n \geq 0 \\ -R\,f(Rx) & , x_n < 0 \end{cases}$$

$$\hat{f}_0(x) = \begin{cases} f_0(x) & , x_n > 0 \\ -f_0(Rx) & , x_n < 0. \end{cases}$$

$$\nabla \tilde{u}(x) = \begin{cases} u(x), & x_n > 0 \\ -R\,\nabla u(Rx) & , x_n < 0. \end{cases}$$

$$\varphi_0(x) := \tfrac{1}{2}\,(\varphi(x) + \varphi(Rx)) \quad \text{even}$$
$$\varphi_1(x) := \tfrac{1}{2}\,(\varphi(x) - \varphi(Rx)) \quad \text{odd.}$$

$$\int_{\mathcal{U}_0} \langle \underbrace{\nabla\tilde{u} - \hat{f}}_{\text{odd}}, \nabla\varphi \rangle - \int_{\mathcal{U}_0} \hat{f}_0\, \varphi$$

$$= \int_{\mathcal{U}_0} \underbrace{\langle \nabla\tilde{u} - \hat{f}, \nabla\varphi_1 \rangle}_{\text{even}} - \int \hat{f}_0\, \varphi_1 \qquad \varphi_1\big|_{\mathcal{U}_0 \cap \partial\Omega} = 0.$$

$$= 2 \int_{\mathcal{U}_0 \cap \Omega} \langle \nabla u - f_i, \nabla\varphi_1 \rangle - \int_{\mathcal{U}_0 \cap \Omega} f_0\, \varphi_1 = 0$$

Case 1

$$\Longrightarrow \quad \|\nabla\tilde{u}\|_{\mathcal{L}^p} \leq C\,\Big(\|\tilde{u}\|_{\mathcal{L}^p} + \sum_{i=0}^{n} \|\hat{f}_i\|_{\mathcal{L}^p}\Big)$$

$$\Downarrow$$

$$2^{1/p}\,\|\nabla u\|_{\mathcal{L}^p} \leq 2^{1/p}\,\Big(\|u\|_{\mathcal{L}^p} + \sum_i \|f_i\|_{\mathcal{L}^p}\Big).$$

<u>Case 3</u>: $x_0 \in \bar{\Omega}$, $a_{ij}(x_0) = \delta_{ij}$

(if $x_0 \in \partial\Omega$ assume Ω is as in Case 2)

Choose $\mathcal{U}_0, c_0$ as in Case 1, 2.
Choose $\varepsilon > 0$ so small that $n^2 c_0 \varepsilon < \frac{1}{2}$
Shrink $\mathcal{U}_0$, if necessary so that

$$|a_{ij}(x) - \delta_{ij}| < \varepsilon \quad \forall x \in \mathcal{U}_0 .$$

$$\Rightarrow \int_\Omega \langle \nabla u, \nabla \varphi \rangle = \int_\Omega \sum_{ij} \frac{\partial u}{\partial x_i} (\delta_{ij} - a_{ij}) \frac{\partial \varphi}{\partial x_j} + \underbrace{\int_\Omega \frac{\partial u}{\partial x_i} a_{ij} \frac{\partial \varphi}{\partial x_j}}_{= \int_\Omega \left(f_0 \varphi + \sum_{i=1}^n f_i \frac{\partial \varphi}{\partial x_i}\right)}$$

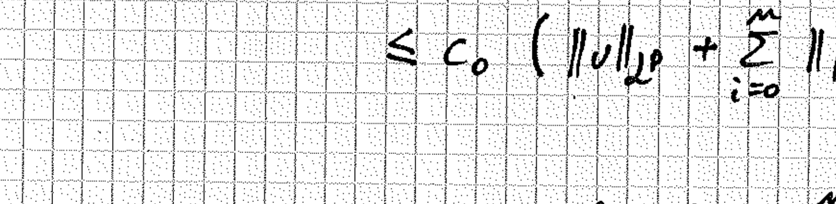
$$= \int_\Omega \sum_{i=1}^n \left(f_i + \sum_j \frac{\partial u}{\partial x_j} (\delta_{ij} - a_{ij})\right) \frac{\partial \varphi}{\partial x_i} + \int_\Omega f_0 \varphi$$

$$\overset{\text{Case 1+2}}{\Longrightarrow} \quad \|\nabla u\|_{L^p} \le c_0 \Big(\|u\|_{L^p} + \|f_0\|_{L^p} + \sum_{i=1}^n \Big\| f_i + \sum_j (\delta_{ij} - a_{ij}) \frac{\partial u}{\partial x_j} \Big\|_{L^p} \Big)$$

$$\le c_0 \Big(\|u\|_{L^p} + \sum_{i=0}^n \|f_i\|_{L^p} \Big) + \underbrace{c_0 n^2 \varepsilon}_{\le 1/2} \|\nabla u\|_{L^p}$$

$$\Rightarrow \quad \|\nabla u\|_{L^p} \le 2 c_0 \Big(\|u\|_{L^p} + \sum_{i=0}^n \|f_i\|_{L^p} \Big).$$

<u>Case 4</u>: $x_0 \in \bar{\Omega}$ arbitrary (interior pt. apply a linear change of var.)



Choose a diffeo (coord chart) $\psi : \underset{x_0}{\overset{\cup}{U_0}} \to \underset{0}{\overset{\cup}{\tilde{U}_0}} \overset{\text{open}}{\subset} \mathbb{R}^n$

1) $\psi(x_0) = 0$
2) $\psi(U_0 \cap \Omega) = \{ \tilde{x} \in \tilde{U}_0 \mid \tilde{x}_n > 0 \}$
3) $\displaystyle\sum_{i,j=1}^n \frac{\partial \psi_k}{\partial x_i}(x_0)\, a_{ij}(x_0)\, \frac{\partial \psi_\ell}{\partial x_j}(x_0) = \delta_{k\ell}\, |\det(d\psi(x_0))|$

⑦

Define : $\tilde{v}, \tilde{\varphi}, \tilde{f}_0 : \tilde{U}_0 \to \mathbb{R}$, $\tilde{f} : \tilde{U}_0 \to \mathbb{R}^n$, $\tilde{a}_{kl} : \tilde{U}_0 \to \mathbb{R}$

by:

$$\tilde{v} \circ \psi = v$$
$$\tilde{\varphi} \circ \psi = \varphi$$
$$\tilde{a}_{kl}(\psi(x)) \, |\det(d\psi(x))| = \sum_{i,j} \frac{\partial \psi_k}{\partial x_i}(x) \, a_{ij}(x) \, \frac{\partial \psi_l}{\partial x_j}(x)$$
$$\tilde{f}_0(\psi(x)) \, |\det(d\psi(x))| := f_0(x)$$
$$\tilde{f}(\psi(x)) \, |\det(d\psi(x))| := d\psi(x) \cdot f(x)$$

Put : $A(x) = [a_{ij}(x)]_{i,j}$ $\quad \tilde{A}(x) = [\tilde{a}_{ij}(x)]_{ij}$

$$\Rightarrow \int_{U_0} \partial_i v \, a_{ij} \, \partial_j \varphi$$
$$= \int_{U_0} \nabla v^T A \nabla \varphi$$
$$= \int_{U_0} \nabla \tilde{v}^T(\psi) \, d\psi A \, d\psi^T \, \nabla \tilde{\varphi}(\psi)$$
$$= \int_{U_0} \nabla \tilde{v}^T(\psi) \, \tilde{A}(\psi) \, \nabla \tilde{\varphi}(\psi) \, |\det(d\psi)|$$
$$= \int_{\tilde{U}_0} (\nabla \tilde{v})^T \tilde{A} \, \nabla \tilde{\varphi}.$$

$$\|$$

$$\int_{U_0} f_0 \varphi + \langle f, \nabla \varphi \rangle$$
$$\|$$
$$\int_{\tilde{U}_0} \tilde{f}_0 \tilde{\varphi} + \langle \tilde{f}, \nabla \tilde{\varphi} \rangle \qquad \forall \, \tilde{\varphi} \in C_0^\infty(\tilde{U}_0 \cap \Omega)$$

By construction, $\tilde{A}(0) = \mathbb{1}$.

$$\overset{\text{Case 3}}{\Longrightarrow} \quad \|\nabla v\|_{L^p} \le \tilde{C}_0 \left(\|\tilde{v}\|_{L^p} + \sum_{i=0}^{n} \|\tilde{f}_i\|_{L^p} \right)$$

As ψ is a diffeo, U_0 bounded, we have:

$$\|\nabla v\|_{L^p} \le c_1 \|\nabla \tilde{v}\|_{L^p}$$
$$\|\tilde{v}\|_{L^p} \le c_2 \|v\|_{L^p}$$
$$\sum_{i=0}^{n} \|\tilde{f}_i\|_{L^p} \le c_3 \sum_{i=0}^{n} \|f_i\|_{L^p} \Longrightarrow \text{Case 4}$$

qed.

Cor 1 $\quad 1<p<\infty, \ \frac{1}{p}+\frac{1}{q}=1.$

Ω, a_{ij} as in Thm 3.

$$B(u,v) := \int_\Omega \frac{\partial u}{\partial x_i} a_{ij} \frac{\partial v}{\partial x_j}.$$

$\Longrightarrow \exists c>0 \ \forall u \in W_0^{1,p}(\Omega)$

$$\|\nabla u\|_{\mathcal{L}^p} \leq c\left(\sup_{0\neq\varphi\in W_0^{1,q}(\Omega)} \frac{|B(u,\varphi)|}{\|\nabla\varphi\|_{\mathcal{L}^q}} + \|u\|_{\mathcal{L}^p}.\right)$$

Proof: $\quad c>0$, const of Thm 3

$Y := \{\nabla\varphi \mid \varphi \in W_0^{1,q}(\Omega)\} \subset \mathcal{L}^q(\Omega,\mathbb{R}^n)$

Fix $u \in W_0^{1,p}(\Omega)$

Define $\Lambda : Y \to \mathbb{R}$

$$\Lambda(\nabla\varphi) := B(u,\varphi) \quad \text{for } \varphi \in W_0^{1,q}(\Omega).$$

$\overset{H-B}{\Longrightarrow}$ Λ extends to a bounded lin. functional on $\mathcal{L}^q(\Omega,\mathbb{R}^n)$

$\overset{(\mathcal{L}^q)^*=\mathcal{L}^p}{\Longrightarrow}$ $\exists f \in \mathcal{L}^p(\Omega,\mathbb{R}^n)$ s.t.

$$\left[\int_\Omega \langle f, \nabla\varphi\rangle = \Lambda(\nabla\varphi) = B(u,\varphi) \quad \forall \varphi \in W_0^{1,q}(\Omega)\right.$$

$$\|f\|_{\mathcal{L}^p} = \|\Lambda|_Y\| = \sup_{\substack{\varphi\neq 0,\\ \varphi\in Y}} \frac{|B(u,\varphi)|}{\|\nabla\varphi\|_{\mathcal{L}^q}}$$

$\Longrightarrow$ u, f satisfy hyp of Thm 3

$$\Longrightarrow \|\nabla u\|_{\mathcal{L}^p} \leq c\left(\|u\|_{\mathcal{L}^p} + \|f\|_{\mathcal{L}^p}\right)$$
$$\leq c\left(\|u\|_{\mathcal{L}^p} + \sup_{\substack{\varphi\neq 0\\ \varphi\in Y}} \frac{|B(u,\varphi)|}{\|\nabla\varphi\|_{\mathcal{L}^q}}\right)$$

qed.

Cor 2

$\mathcal{L}: W_0^{1,p}(\Omega) \to W^{-1,p}(\Omega)$ has a closed image and a finite dim'l kernel.

Pf: Cor 1 + Chap 4 FA I.
$W_0^{1,p}(\Omega) \to L^p(\Omega)$ is compact.
qed

Lecture 3 28/04/2016

$\Omega \subset \mathbb{R}^n$ open, bounded, $\partial\Omega$ smooth
$a_{ij} \in C^1(\bar{\Omega})$
$\sum_{i,j} a_{ij}(x)\, \xi_i \xi_j \geq \delta |\xi|^2 \quad \forall x \in \bar{\Omega} \quad \forall \xi \in \mathbb{R}^n$
$\mathcal{L}u = -\sum_{i,j} \frac{\partial}{\partial x_i}\left(a_{ij} \frac{\partial}{\partial x_j} u\right)$

$$B(u,v) = \sum_{i,j} \int_\Omega \frac{\partial u}{\partial x_i}\, a_{ij}\, \frac{\partial v}{\partial x_j}$$

Thm 4 $1 < p < \infty$.
Let $u \in W_0^{1,p}(\Omega)$, $f \in L^p(\Omega)$ s.t.
$B(u,\varphi) = \int_\Omega f\varphi \quad \forall \varphi \in C_0^\infty(\Omega)$
$\Rightarrow u \in W^{2,p}(\Omega)$, $\mathcal{L}u = f$.

Thm 3 and 4 $\Rightarrow$ Thm 2:

Step 1: $\mathcal{L}: W_0^{1,p}(\Omega) \to W^{-1,p}(\Omega)$ has a closed image and a finite dim'l kernel.

Proof: By Thm 3, $\exists c > 0 \ \forall u \in W_0^{1,p}(\Omega)$
$\|\nabla u\|_{L^p} \leq c\left(\|\mathcal{L}u\|_{W^{-1,p}} + \|u\|_{L^p}\right)$
The inclusion $W_0^{1,p}(\Omega) \to L^p(\Omega)$ is a cpct op. by Rellich's thm (Ch. II, Thm 5). Now we use the main Fredholm lemma in FA I, Ch IV.

<u>Step 2</u>: $\mathcal{L}: W^{1,p}(\Omega) \to W^{-1,p}(\Omega)$ is injective.

<u>Proof</u>:

<u>$p \geq 2$</u>: $W_0^{1,p}(\Omega) \subset W_0^{1,2}(\Omega)$ ✓

<u>$p < 2$</u>: Let $u \in W_0^{1,p}(\Omega)$ s.t. $B(u, \varphi) = 0 \quad \forall \varphi \in C_0^\infty(\Omega)$.

$$\overset{\text{Thm 4}}{\Longrightarrow} \quad u \in W^{2,p}(\Omega) \subset W^{1,p_1}(\Omega) \quad , \; p_1 := \frac{np}{n-p} > p$$

$$\Rightarrow \quad u \in W_0^{1,p_1}(\Omega) \Rightarrow u \in W^{1,p_2} \quad , \; p_2 := \frac{np_1}{n-p_1} > p_1.$$

$$\underset{k \geq 2.}{\overset{\text{Induction}}{\Longrightarrow}} \quad u \in W_0^{1,p_k}(\Omega) \qquad p_k := \frac{n p_{k-1}}{n - p_{k-1}} > p_{k-1}.$$

Exercise: $p_k \geq \frac{n}{n-k}$

So for $k \geq \frac{n}{2}$, we have $p_k > 2$.

<u>Step 3</u>: $\mathcal{L}: W_0^{1,p}(\Omega) \to W^{-1,p}(\Omega)$ is surjective.

$$\begin{array}{ccccc} \mathcal{L}^* & : & \left(W^{-1,p}(\Omega)\right)^* & \to & \left(W_0^{1,p}(\Omega)\right)^* \quad \text{inj.} \\ \parallel & & \parallel & & \parallel \\ \mathcal{L} & & W_0^{1,q}(\Omega) & & W^{-1,q}(\Omega) \end{array}$$

$\Rightarrow$ $\mathcal{L}^*$ has a dense image $\forall q > 1$

$\Rightarrow$ $\mathcal{L}: W_0^{1,p}(\Omega) \to W^{-1,p}(\Omega)$ has a dense closed image $\forall p > 1$

$\Rightarrow$ surjective

qed.

<u>Thm 2 and 4 $\Rightarrow$ Thm 1</u>

$\mathcal{L}: W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \to L^p(\Omega)$ is bijective.

<u>injective:</u>

$$\mathcal{L}u = 0 \Rightarrow u \in W_0^{1,p}(\Omega) \;,\; \underbrace{B(u, \varphi)}_{= \int_\Omega (\mathcal{L}u)\varphi} = 0 \quad \forall \varphi \in C_0^\infty(\Omega)$$

$$\overset{\text{Thm 2}}{\Longrightarrow} \quad u = 0 \quad , \text{ as } \mathcal{L}: W_0^{1,p}(\Omega) \to W^{-1,p}(\Omega) \text{ is injective.}$$

<u>surjective:</u>

Let $f \in L^p(\Omega)$. Define $\Phi_f \in W^{-1,p}(\Omega) = \left(W_0^{1,q}(\Omega)\right)^*$ by

$$\Phi_f(\varphi) := \int_\Omega f \varphi \quad \text{for } \varphi \in W_0^{1,q}(\Omega)$$

$$\overset{\text{Thm 2}}{\Longrightarrow} \quad \exists\, u \in W_0^{1,p}(\Omega) \text{ s.t. } \quad \mathcal{L}u = \Phi_f$$

$$\text{i.e.} \quad B(u,\varphi) = \Phi_f(\varphi) = \int_\Omega f\varphi \quad \forall \varphi \in C_0^\infty(\Omega)$$

$$\overset{\text{Thm 4}}{\Longrightarrow} \quad u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \text{ and } \mathcal{L}u = f \qquad \text{qed.}$$

Proof of Thm 4:

Difference quotient method:

$$1 \le k \le n, \quad e_k = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow k \in \mathbb{R}^n$$

$$v \in W^{1,p}(\mathbb{R}^n), \quad h \in \mathbb{R} \setminus \{0\}$$

$$v^h(x) := \frac{v(x+he_k) - v(x)}{h}$$

Fact 1: $\|v^h\|_{L^p} \le \|\partial_k v\|_{L^p} \quad \forall h \in \mathbb{R}\setminus\{0\}$

Fact 2: $v \in L^p(\mathbb{R}^n)$ such that $\sup_{h \neq 0} \|v^h\|_{L^p} < \infty$

$$\Longrightarrow \exists\, v_k \in L^p(\mathbb{R}^n) \quad \forall \varphi \in C_0^\infty(\mathbb{R}^n)$$

$$-\int_{\mathbb{R}^n} v \frac{\partial \varphi}{\partial x_k} = \int_{\mathbb{R}^n} v_k \varphi.$$

Fact 3: $(fg)^h(x) = f(x)g^h(x) + f^h(x)g(x_h), \quad x_h = x + he_k$

Fact 4: $\partial_i(v^h) = (\partial_i v)^h.$

Assume $u \in W_0^{1,p}(\Omega)$, $f \in L^p(\Omega)$, $B(u,\varphi) = \int_\Omega f\varphi \ \forall \varphi \in C_0^\infty(\Omega)$

Claim: If $\rho \in C_0^\infty(\Omega)$, $k \in \{1,\dots,n\}$, $v := \rho u$,

$$v^h(x) := \frac{v(x+he_k) - v(x)}{h}$$

$$\Longrightarrow \exists\, c > 0, \forall h \in \mathbb{R}\setminus\{0\}, \forall \varphi \in C_0^\infty(\Omega) \quad |B(\varphi, v^h)| \le c\, \|\nabla\varphi\|_{L^q}$$

Claim $\Rightarrow v \in W^{1,p}_{loc}(\Omega)$

$$\|(\nabla v)^h\|_{L^p} \overset{\text{Fact 4}}{=} \|\nabla(v^h)\|_{L^p}$$

$$\overset{\text{Thm 3}}{\leq} c_1 \left(\|v^h\|_{L^p} + \sup_{0 \neq \varphi \in C_0^\infty(\Omega)} \frac{|B(v^h, \varphi)|}{\|\nabla \varphi\|_{L^q}} \right)$$

$$\underset{\text{Claim}}{\overset{\text{Fact 1}}{\leq}} c_1 \left(\|\partial_i v\|_{L^p} + c \right)$$

$$\overset{\text{Fact 2}}{\Longrightarrow} \underset{k=1,\dots,n}{\partial_k \nabla v \in L^p(\Omega)} \Rightarrow \nabla v \in W^{1,p}(\Omega)$$

$$\Rightarrow \rho v \in W^{2,p}(\Omega) \quad \forall \rho \in C_0^\infty(\Omega)$$

$$\Rightarrow v \in W^{2,p}_{loc}(\Omega).$$

Proof of Claim:

$$B(v^h, \varphi) = \sum_{i,j} \int_\Omega \partial_i (\rho v)^h \, a_{ij} \, \partial_j \varphi$$

$\frac{1}{p} + \frac{1}{q} = 1$

$$\overset{\text{Fact 4}}{=} \sum_{i,j} \int_\Omega (\partial_i \rho v + \rho \partial_i v)^h \, a_{ij} \, \partial_j \varphi$$

$$\overset{\text{Fact 3}}{=} \sum_{i,j} \int_\Omega \underbrace{(\partial_i \rho)^h}_{\text{bnded}} \underbrace{v(\cdot + h e_i)}_{L^p} \underbrace{a_{ij}}_{\text{bnded}} \underbrace{\partial_j \varphi}_{L^q}$$

$$+ \sum_{i,j} \int_\Omega \underbrace{(\partial_i \rho)}_{\text{bnded}} \underbrace{v^h}_{L^p} \underbrace{a_{ij}}_{\text{bnded}} \underbrace{\partial_j \varphi}_{L^q} \quad \Bigg\} \leq c \|\nabla \varphi\|_{L^q}$$

$$I =: \Big\{ + \sum_{i,j} \int_\Omega (\rho \partial_i v)^h \, a_{ij} \, \partial_j \varphi$$

$$I = \sum_{i,j} \int_\Omega (\rho \, \partial_i v \, a_{ij})^h \, \partial_j \varphi \quad \Big\} =: II$$

$$- \sum_{i,j} \int_\Omega \underbrace{(\rho \, \partial_i v)}_{L^p} (\cdot + h e_k) \underbrace{a_{ij}^h}_{\substack{\text{bnded} \\ \text{as } a_{ij} \in C^1(\bar\Omega)}} \underbrace{\partial_j \varphi}_{L^q} \quad \Big\} \leq c \|\nabla \varphi\|$$

$$\text{II} = + \sum_{i,j} \int_\Omega \rho (\partial_i u) a_{ij} (\partial_j \varphi)^{-h}$$

$$\overset{\text{Fact 4}}{=} + \sum_{i,j} \int_\Omega (\partial_i u) a_{ij} \partial_j (\rho \varphi)^{-h} \quad \Big\} = \text{III}.$$

$$- \sum_{i,j} \int_\Omega \underbrace{(\partial_i u)}_{L^p} \underbrace{a_{ij}(\partial_j \rho)}_{\text{bounded}} \underbrace{\varphi^{-h}}_{L^q} \quad \Big\} \leq c \|\nabla \varphi\|_{L^q}$$

$$\text{III} = B(u, + \rho \varphi^{-h})$$

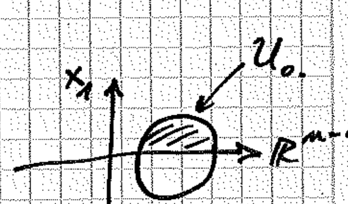
$$\overset{\text{Assumption}}{=} + \int_\Omega (f \rho \varphi^{-h})$$

$$|\text{III}| \overset{\text{Hölder}}{\leq} \|\rho f\|_{L^p} \|\varphi^{-h}\|_{L^q}$$

$$\overset{\text{Fact 1}}{\leq} \|\rho f\|_{L^p} \|\nabla \varphi\|_{L^q} \qquad \checkmark$$

<u>Boundary regularity:</u>

Let $x_0 \in \partial\Omega$.



<u>Case 1:</u> $x_0 \in \mathbb{R}^{n-1} \times \{0\}$

Assume $\Omega \cap B_{2\varepsilon}(x_0) = \{x \in B_{2\varepsilon}(x_0) \mid x_n > 0\}$

$$U_0 := \Omega \cap B_\varepsilon(x_0) = \{x \in B_\varepsilon(x_0) \mid x_n > 0\}$$

Choose $\rho \in C_0^\infty(\mathbb{R}^n)$. $\operatorname{supp}(\rho) \subset B_{2\varepsilon}(x_0)$

$$\rho \equiv 1 \text{ on } B_\varepsilon(x_0)$$

$v := \rho u \quad , \quad h \neq 0$

$$v^h(x) := \frac{v(x + h e_k) - v(x)}{h} \quad , \quad \begin{matrix} x \in B_{2\varepsilon}(x_0) \\ x_n \geq 0. \end{matrix}$$

$k = 1, \dots, n-1 \quad \leftarrow$ well-defined for $k \neq n$.

$\Rightarrow v^h \in W_0^{1,p}(\Omega)$

as above: $\sup \| \nabla v^k \|_{\mathcal{L}^p} < \infty$

Fact 2

$\Rightarrow \quad \partial_k \nabla v \in \mathcal{L}^p(B_{2\varepsilon}(x_0) \cap \Omega) \qquad \forall\, k = 1, \dots, n-1.$

$\Rightarrow \quad \partial_k u \in W^{1,p}(\mathcal{U}_0) \qquad k = 1, \dots, n-1.$

$\Rightarrow \quad \partial_i \partial_j u \in \mathcal{L}^p(\mathcal{U}_0) \qquad i,j = 1, \dots, n, \ (i,j) \neq (n,n)$

Why is $\frac{\partial^2 u}{\partial x_0^2} \Big|_{\mathcal{U}_0} \in \mathcal{L}^p(\mathcal{U}_0)$?

<u>Answer</u>: use the eq'n:

$$f = \mathcal{L} u = - \sum_{i,j} \partial_i (a_{ij} \partial_j u)$$

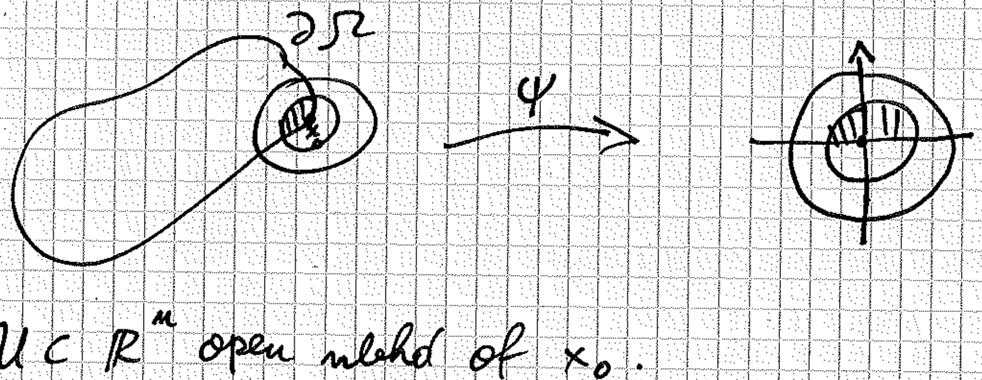
$$= g - a_{nn} \partial_n \partial_n u$$

$$g := \sum_{(i,j) \neq (n,n)} \partial_i (a_{ij} \partial_j u) + (\partial_n a_{nn}) \partial_n u \in \mathcal{L}^p(\mathcal{U}_0)$$

$\Rightarrow \quad \partial_n \partial_n u = \frac{g - f}{a_{nn}} \in \mathcal{L}^p(\mathcal{U}_0) \qquad$ ($a_{nn} > \delta$ by ellipticity)

$\Rightarrow \quad u|_{\mathcal{U}_0} \in W^{2,p}(\mathcal{U}_0)$

<u>Case 2</u>: $x_0 \in \partial\Omega$ arbitrary



$\mathcal{U} \subset \mathbb{R}^n$ open nbhd of x_0.

$\Psi: \mathcal{U} \to B_{2\varepsilon}(0)$ diffeo

$\Psi(\mathcal{U} \cap \Omega) = \{ x \in B_{2\varepsilon} \mid x_n > 0 \}$

$\mathcal{U}_0 := \Psi^{-1}(\underbrace{\{ x \in B_\varepsilon \mid x_n > 0 \}}_{V_0})$

Case 1

$\Rightarrow \quad u \circ \Psi^{-1} \in W^{2,p}(V_0) \Rightarrow u|_{\mathcal{U}_0} \in W^{2,p}(\mathcal{U}_0).$ qed

Thm 5 (Higher regularity)

$1 < p < \infty$, $k \in \mathbb{N}$

$\Omega, a_{ij}, \mathcal{L}, B$ as in Thm 4

$a_{ij} \in C^{k+1}(\overline{\Omega})$

(i) If $u \in W_0^{1,p}(\Omega)$, $f \in W^{k,p}(\Omega)$

$$B(u,\varphi) = \int_\Omega f\varphi \quad \forall \varphi \in C_0^\infty(\Omega)$$

$$\Rightarrow u \in W^{k+2,p}(\Omega), \ \mathcal{L}u = f.$$

(ii) The operator

$$\mathcal{L} : W^{k+2,p}(\Omega) \cap W_0^{1,p}(\Omega) \to W^{k,p}(\Omega)$$

is bijective

(iii) $\exists C > 0 \quad \forall u \in W^{k+2,p}(\Omega) \cap W_0^{1,p}(\Omega)$

$$\|u\|_{W^{k+2,p}} \le C \|\mathcal{L}u\|_{W^{k,p}}$$

Rmk:

(i) $k = 0$ is Thm 4 + Induction

(ii) follows directly from (i) and Thm 1.

(iii) follows directly from (ii) and open mapping thm.

Lecture 4 02/05/2016

Proof of Thm 5 (i) Ind. Step

$k \ge 1$ Assume (i) holds for $k-1$

$u \in W_0^{1,p}(\Omega)$, $f \in W^{k,p}(\Omega)$ s.t.

$B(u,\varphi) = \int_\Omega f\varphi \quad \forall \varphi \in C_0^\infty(\Omega)$

$\overset{\text{Ind. Hypo.}}{\Longrightarrow} u \in W^{k+1,p}(\Omega)$ and $\mathcal{L}u = f$.

To show: $u \in W^{k+2,p}(\Omega)$

Idea of proof:

$$\mathcal{L}\,\frac{\partial u}{\partial x_\nu} = \underbrace{\mathcal{L}\frac{\partial u}{\partial x_\nu} - \frac{\partial}{\partial x_\nu}(\mathcal{L}u)}_{\substack{2^{nd}\text{ order operator}\\ \in W^{k-1,p}}} + \underbrace{\frac{\partial f}{\partial x_\nu}}_{\substack{\cap\\ W^{k-1,p}}} = g.$$

$$\overset{\text{Ind. hyp.}}{\Longrightarrow} \quad \frac{\partial u}{\partial x_\nu} \in W^{k+1,p}_{loc}(\Omega), \quad \nu = 1, \dots, n$$

Problem: $\quad \left.\frac{\partial u}{\partial x_\nu}\right|_{\partial\Omega} \neq 0$

<u>Boundary regularity</u>:

Method 1: proceed as in Case 1 + Case 2 as before in Thm 4.

Method 2:

$$\mathrm{Vect}(\Omega, \partial\Omega) = \left\{ g \in C^\infty(\Omega, \mathbb{R}^n) \;\middle|\; \begin{array}{l} g(x) \in T_x\partial\Omega \\ \forall x \in \partial\Omega \end{array} \right\}$$

$$\Longrightarrow \quad \langle \nabla u, g \rangle \in W^{1,p}_0(\Omega)$$

$$P_g(u) := \mathcal{L}\langle \nabla u, g\rangle - \langle \nabla(\mathcal{L}u), g\rangle.$$

$$= \sum_{i,j,\nu} \Big[g_\nu \frac{\partial}{\partial x_i}\Big(\frac{\partial a_{ij}}{\partial x_\nu} \frac{\partial u}{\partial x_j}\Big) - \frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial g_\nu}{\partial x_i}\frac{\partial u}{\partial x_\nu}\Big) - \frac{\partial g_\nu}{\partial x_i} a_{ij} \frac{\partial^2 u}{\partial x_j \partial x_\nu}\Big] \in W^{k-1,p}(\Omega)$$

because $u \in W^{k+1,p}(\Omega)$.

$\longrightarrow$ cf. p. 15

$$\boxed{B(\langle \nabla u, g\rangle, \varphi) = \int_\Omega \underbrace{(\langle \nabla f, g\rangle + P_g(u))}_{W^{k-1,p}(\Omega)} \varphi}$$

$$\overset{\text{Ind.}}{\underset{\text{Hyp.}}{\Longrightarrow}} \begin{cases} \langle \nabla u, g\rangle \in W^{k+1,p}(\Omega) & \forall\, g \in \mathrm{Vect}(\Omega, \partial\Omega) \\ \mathcal{L}u = f \in W^{k,p}(\Omega) \end{cases}$$

$$\overset{\text{Excercise}}{\Longrightarrow} \quad u \in W^{k+2,p}(\Omega)$$

qed.

Lemma 3: The space $\{\varphi \in C^\infty(\bar\Omega) \mid \varphi|_{\partial\Omega} = 0\}$ is dense in $W^{k,p}(\Omega) \cap W_0^{1,p}(\Omega)$ for every integer $k \geq 2$.

Proof: Let $u \in W^{k,p}(\Omega) \cap W_0^{1,p}(\Omega)$.

Define $f := \Delta u \in W^{k-2,p}(\Omega)$

$\overset{\text{Ch II}}{\Longrightarrow}$ $\exists$ sequence $f_i \in C^\infty(\bar\Omega)$ s.t.

$$\lim_{i\to\infty} \|f - f_i\|_{W^{k-2,p}} = 0.$$

$\overset{\text{Thm 5}}{\Longrightarrow}$ $\exists!$ sequence $u_i \in W^{k,p}(\Omega) \cap W_0^{1,p}(\Omega)$ s. that $\Delta u_i = f_i$.

$\overset{\text{Thm 5}}{\Longrightarrow}$ $\|u - u_i\|_{W^{k,p}} \leq c\|\Delta(u-u_i)\|_{W^{k-2,p}} = c\|f - f_i\|_{W^{k-2,p}} \xrightarrow{i\to\infty} 0$

Also $u_i \in \bigcap_{k=1}^{\infty} W^{k,p}(\Omega) = C^\infty(\bar\Omega)$, $u_i|_{\partial\Omega} = 0$ qed.

$\Omega \subset \mathbb{R}^n$ open, bnded smooth bndry

$a_{ij} \in C^2(\bar\Omega)$, $i,j = 1,\dots,n$

$\sum a_{ij}(x)\xi_i\xi_j \geq \delta|\xi|^2 \quad \forall x \in \bar\Omega \ \forall \xi \in \mathbb{R}^n.$

$b_i \in C^1(\bar\Omega)$ $i = 1,\dots,n$

$c \in C^0(\bar\Omega)$

$$Lu := \sum_{i,j=1}^n a_{ij}\partial_i\partial_j u + \sum_{i=1}^n b_i \partial_i u + cu.$$

$$L^*v := \sum_{i,j=1}^n \partial_i\partial_j(a_{ij}v) - \sum_{i=1}^n \partial_i(b_i v) + cv$$

Exercise:

$$\boxed{\int_\Omega (Lu)\, v = \int_\Omega u\,(L^*v)}$$

$\forall u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$

$\forall v \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$ $\quad \frac{1}{q} + \frac{1}{p} = 1$, $1 < p < \infty$.

Thm 6: $1 < p, q < \infty$, $\frac{1}{q} + \frac{1}{p} = 1$, $\Omega, a_{ij}, b_i, c, L, L^*$ as above.

(i) $$\left[\begin{array}{l} u \in L^p(\Omega), \quad \Phi \in W^{-1,p}(\Omega) = (W_0^{1,q}(\Omega))^* \\ \int_\Omega u L^* \varphi = \Phi(\varphi) \\ \forall \varphi \in C^\infty(\bar\Omega), \ \varphi|_{\partial\Omega} = 0 \end{array}\right\} \Rightarrow u \in W_0^{1,p}(\Omega)$$

and $\exists c > 0 \ \forall u \in W_0^{1,p}(\Omega)$.

$$\left[\|u\|_{W^{1,p}} \le c \left(\|Lu\|_{W^{-1,p}} + \|u\|_{L^p} \right) \right.$$

(ii) Let $k \in \mathbb{N}_0$ and assume
$a_{ij} \in C^k(\bar\Omega) \cap C^2(\bar\Omega)$, $b_i \in C^k(\bar\Omega) \cap C^1(\bar\Omega)$, $c \in C^k(\bar\Omega)$

$$\left[\begin{array}{l} u \in L^p(\Omega), \quad f \in W^{k,p}(\Omega) \\ \int_\Omega u L^* \varphi = \int_\Omega f \varphi \\ \quad \forall \varphi \in C^\infty(\bar\Omega), \ \varphi|_{\partial\Omega} = 0 \end{array}\right\} \Rightarrow \begin{array}{l} u \in W^{k+2,p}(\Omega) \cap W_0^{1,p}(\Omega) \\ Lu = f \end{array}$$

and $\exists c > 0 \ \forall u \in W^{k+2,p} \cap W_0^{1,p}(\Omega)$:

$$\left[\|u\|_{W^{k+2,p}} \le c \left(\|Lu\|_{W^{k,p}} + \|u\|_{L^p} \right). \right.$$

Proof:

$$L_0 \overset{!}{:=} -\sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial}{\partial x_j} \right)$$

$$B_0(u,v) := \int_\Omega \sum_{i,j} \frac{\partial u}{\partial x_i} a_{ij} \frac{\partial v}{\partial x_j}$$

1. $$Pu := \sum_i \left(b_i - \sum_j \partial_j a_{ij} \right) \partial_i u + cu$$
$$= Lu + L_0 u$$

2. $\exists c > 0 \quad \forall \varphi \in W_0^{1,q}(\Omega)$

$$\|P^* \varphi\|_{L^q} \le c \, \|\nabla \varphi\|_{L^q}$$

$$P^* \varphi = -\sum_i \partial_i \left(\left(b_i - \sum_j \partial_j a_{ij} \right) \varphi \right) + c\varphi.$$

3. Define $\Psi \in W^{-1,p}(\Omega)$ by $\Psi(\varphi) := \int_\Omega u P^* \varphi$

$$\boxed{\|\Psi\|_{W^{-1,p}} = \sup_{0 \ne \varphi \in C_0^\infty(\Omega)} \frac{\left| \int_\Omega u P^* \varphi \right|}{\|\nabla \varphi\|_{L^q}} \overset{\text{Hölder}}{\le} c \, \|u\|_{L^p}}$$

4. $\mathcal{L}_0 : W_0^{1,p}(\Omega) \to W^{-1,p}(\Omega)$ bijective (Thm 2)

$\Rightarrow \exists!\ v \in W_0^{1,p}(\Omega)$ s.t.

$$B_0(v,\varphi) = \Psi(\varphi) - \Phi(\varphi) \quad \forall \varphi \in W_0^{1,q}(\Omega)$$

5. $\forall \varphi \in C^\infty(\bar\Omega)$, $\varphi|_{\partial\Omega} = 0$, we have:

$$\int_\Omega (u-v)\mathcal{L}_0\varphi = \int_\Omega u\,(P^*\varphi - \mathcal{L}^*\varphi) - B_0(v,\varphi)$$

$$= \Psi(\varphi) - \Phi(\varphi) - B_0(v,\varphi) = 0.$$

$\overset{\text{Lemma 3}}{\Longrightarrow} \int_\Omega (u-v)\mathcal{L}_0\varphi = 0 \quad \forall \varphi \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$

$\overset{\text{Thm 1}}{\Longrightarrow} \int_\Omega (u-v) f = 0 \quad \forall f \in L^q(\Omega)$

$\Rightarrow u = v \in W_0^{1,p}(\Omega)$

$$\boxed{\|u\|_{W^{1,p}} \le C_0 \|\mathcal{L}_0 u\|_{W^{-1,p}} \le C_0 \big(\|\mathcal{L}u\|_{W^{-1,p}} + \underbrace{\|Pu\|_{W^{-1,p}}}_{\le C\|u\|_{L^p}}\big)}$$

(ii) $u \in L^p(\Omega)$, $f \in W^{k,p}(\Omega)$

$$\int_\Omega u\mathcal{L}^*\varphi = \int_\Omega f\varphi \quad \forall \varphi \in C^\infty(\bar\Omega),\ \varphi|_{\partial\Omega} = 0$$

$\overset{(i)}{\Longrightarrow} u \in W_0^{1,p}(\Omega)$

$$B_0(u,\varphi) = \int_\Omega u\mathcal{L}_0\varphi = \int_\Omega u\,(P^*\varphi - \mathcal{L}^*\varphi)$$

$$= \int_\Omega \underbrace{(Pu - f)}_{\in L^p}\varphi$$

$\overset{\text{Thm 4}}{\Longrightarrow} u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ and $\mathcal{L}_0 u = Pu - f \in W^{1,p}(\Omega)$

$\overset{\text{Thm 5}}{\Longrightarrow} u \in W^{3,p}(\Omega) \cap W_0^{1,p}(\Omega)$ and $\mathcal{L}_0 u \in W^{2,p}(\Omega)$ (if $k \ge 2$)

$\vdots$

$\overset{\text{Thm 5}}{\Longrightarrow}$ induction $u \in W^{k+2,p}(\Omega)$, $\mathcal{L}u = Pu - \mathcal{L}_0 u = f$.

$$\|u\|_{W^{k+2,p}} \overset{\text{Thm 5}}{\le} C_0 \|\mathcal{L}_0 u\|_{W^{k,p}}$$

$$\le C_0 \big(\|\mathcal{L}u\|_{W^{k,p}} + \|Pu\|_{W^{k,p}}\big)$$

$$\le C_1 \big(\|\mathcal{L}u\|_{W^{k,p}} + \|u\|_{W^{k-1,p}}\big)$$

$$\underset{\text{Nirenberg Chap 2.}}{\overset{\text{Gagliardo-}}{\le}} C_2 \big(\|\mathcal{L}u\|_{W^{k,p}} + \|u\|_{L^p}^{\frac{1}{k+2}} \|u\|_{W^{k+2,p}}^{\frac{k+1}{k+2}}$$

Now use $ab \le \frac{1}{r} a^r + \frac{1}{s} b^s$, $\frac{1}{r} + \frac{1}{s} = 1$, $r, s \ge 1$.

$$\|u\|_{W^{k+2,p}} \le c_2 \|\mathcal{L}u\|_{W^{k,p}} + \frac{c_2^{k+2}}{k+2} \|u\|_{\mathcal{L}^p} + \frac{k+1}{k+2} \|u\|_{W^{k+2,p}}$$

$$\Rightarrow \frac{1}{k+2} \|u\|_{W^{k+2,p}} \le c_2 \|\mathcal{L}u\|_{W^{k,p}} + \frac{c_2^{k+2}}{k+2} \|u\|_{\mathcal{L}^p}$$

qed.

<u>Excercise:</u> $\mathcal{L}: W^{k+2,p}(\Omega) \cap W_0^{1,p}(\Omega) \to W^{k,p}(\Omega)$ is a Fredholm operator of index zero.

<u>Thm 7</u> (Local Regularity)

$1 < p < \infty$, $k \in \mathbb{N}_0$

$\Omega, \mathcal{L}, \mathcal{L}^*$ as in Thm 6

$a_{ij} \in C^k(\bar\Omega) \cap C^2(\bar\Omega)$, $b_i \in C^k(\bar\Omega) \cap C^1(\bar\Omega)$, $c \in C^k(\bar\Omega)$

(i) $u \in \mathcal{L}^p_{loc}(\Omega)$, $f \in W^{k,p}_{loc}(\Omega)$

$$\int_\Omega u \mathcal{L}^* \varphi = \int_\Omega f \varphi \quad \forall \varphi \in C_0^\infty(\Omega)$$

$$\Rightarrow u \in W^{k+2,p}_{loc}(\Omega) \text{ and } \mathcal{L}u = f.$$

(ii) $\forall$ cpct set $K \subset \Omega$ $\forall$ integer $k \ge 1$ $\exists c > 0$ $\forall u \in W^{k-2,p}_{loc}(\Omega)$.

$$\|u\|_{W^{k+2,p}(K)} \le c \left(\|\mathcal{L}u\|_{W^{k,p}(\Omega)} + \|u\|_{\mathcal{L}^p(\Omega)} \right)$$

<u>Proof:</u> (i) Induction:

<u>$k=0$:</u> Choose $\rho \in C_0^\infty(\Omega)$, $\rho|_K \equiv 1$

$\beta \in C_0^\infty(\Omega)$, $\beta|_{supp(\rho)} \equiv 1$

So $\beta \cdot \rho = \rho$.

Define $\Phi \in W^{-1,p}(\Omega)$ by

$$\Phi(\varphi) = \int_\Omega u \left(\rho \mathcal{L}^* \varphi - \mathcal{L}^*(\rho\varphi)\right) + \int_\Omega f \rho \varphi, \quad \varphi \in W_0^{1,q}(\Omega)$$

(14)

Note:

1) $\| \rho \mathcal{L}^* \varphi - \mathcal{L}^*(\rho\varphi) \|_{L^q} \le c \|\nabla \varphi\|_{L^q}$

2) $\int_\Omega \rho u \mathcal{L}^* \varphi = \int_\Omega u \left(\rho \mathcal{L}^* \varphi - \mathcal{L}^*(\rho\varphi) \right) + \int_\Omega f \varphi \rho$
$= \Phi(\varphi).$

3) $\varphi \in C^\infty(\bar\Omega)$, $\varphi|_{\partial\Omega} = 0$

$\Rightarrow \beta\varphi \in C_0^\infty(\Omega)$

$$\Rightarrow \int_\Omega \rho u \mathcal{L}^*(\beta\varphi) = \Phi(\beta\varphi)$$

$$\int_\Omega \rho u \mathcal{L}^*(\varphi) \quad = \quad \Phi(\varphi)$$

$\overset{\text{Thm 6(i)}}{\Rightarrow} \rho u \in W_0^{1,p}(\Omega)$

$$\Phi(\varphi) = \int_\Omega \underbrace{f\rho}_{\in L^p} \varphi + \int_\Omega \underbrace{(\mathcal{L}(\rho u) - \rho \mathcal{L} u)}_{\in L^p} \varphi$$

$\Rightarrow |\Phi(\varphi)| \le c \|\varphi\|_{L^q}$

$\Rightarrow \Phi \in (L^q)^* \Rightarrow \exists g \in L^p : \Phi(\varphi) = \int_\Omega g\varphi.$

$\overset{\text{Thm 6(ii)}}{\Rightarrow} \rho u \in W^{2,p}(\Omega) \quad \forall \rho \in C_0^\infty(\Omega)$

$\Rightarrow u \in W^{2,p}_{loc}(\Omega)$, $\mathcal{L} u = f.$

Induction step $k \ge 1$:

Assume (i) holds for $k-1$.

$u \in L^p_{loc}$, $f \in W^{k,p}_{loc}$, $\int_\Omega u \mathcal{L}^* \varphi = \int_\Omega f \varphi$
$\forall \varphi \in C_0^\infty(\Omega)$

$\overset{\text{Ind Hyp.}}{\Rightarrow} u \in W^{k+1}_{loc}(\Omega).$

$$\mathcal{L} \partial_\nu u = \underbrace{(\mathcal{L} \partial_\nu u - \partial_\nu(\mathcal{L} u)) + \partial_\nu f}_{\text{Exercise: } \in W^{k-1,p}_{loc}(\Omega)}$$

$\overset{\text{Ind. hyp.}}{\Rightarrow} \partial_\nu u \in W^{k+1,p}_{loc}(\Omega) \quad \nu = 1, \dots, n$

$\Rightarrow u \in W^{k+2,p}_{loc}(\Omega).$

For (ii), $k=-1$.

$\rho \in C_0^\infty(\Omega)$, $\rho : \Omega \to [0,1]$, $\rho|_K \equiv 1$.

$$\|u\|_{W^{1,p}(K)} = \|\rho u\|_{W^{1,p}(K)} \le \|\rho u\|_{W^{1,p}(\Omega)}$$

$$\overset{\text{Thm 6}}{\le} c\left(\|\mathcal{L}(\rho u)\|_{W^{-1,p}(\Omega)} + \|\rho u\|_{L^p(\Omega)}\right)$$

$$\le c\,\Big(\underbrace{\|\rho(\mathcal{L}u)\|_{W^{-1,p}(\Omega)}}_{\le c'\|\mathcal{L}u\|_{W^{-1,p}(\Omega)}} + c\,\underbrace{\|\mathcal{L}(\rho u) - \rho(\mathcal{L}u)\|_{W^{-1,p}(\Omega)}}_{\le c'\|u\|_{L^p}} + c\|u\|_{L^p(\Omega)}$$

cf. p. 15

Ind Step $k \ge 0$:

Assume the estimate holds for $k-1$.

$u \in W^{k+2,p}_{loc}(\Omega)$, $K \subset \Omega$ cpct.

Choose $\rho \in C_0^\infty(\Omega)$, $\rho|_K \equiv 1$, $K' = supp(\rho)$.

$\overset{\text{Ind hypo}}{\Longrightarrow}$ 1. $\exists\, c' > 0 \;\; \forall u \in W^{k+1}_{loc}(\Omega)$

(i) $\boxed{\|u\|_{W^{k+1,p}(K)} \le c'\left(\|\mathcal{L}u\|_{W^{k-1,p}(\Omega)} + \|u\|_{L^p(\Omega)}\right)}$

2. $\exists\, c'' > 0$, $\forall u \in W^{k+1,p}_{loc}(\Omega)$:

(ii) $\boxed{\|\rho(\mathcal{L}u) - \mathcal{L}(\rho u)\|_{W^{k,p}(\Omega)} \le c''\|u\|_{W^{k+1,p}(K')}}$

3. $P = \mathcal{L} + \mathcal{L}_0$ 1^{st} order op.

$\exists\, c''' > 0 \;\; \forall u \in W^{k+1,p}_{loc}(\Omega)$

(iii) $\boxed{\|P(\rho u)\|_{W^{k,p}(\Omega)} \le c'''\|u\|_{W^{k+1,p}(K')}}$

$$\Rightarrow \|u\|_{W^{k+2,p}(K)} \le \|\rho u\|_{W^{k+2,p}(\Omega)}$$

$$\overset{\text{Thm 5}}{\le} c_0\|\mathcal{L}_0(\rho u)\|_{W^{k,p}(\Omega)}$$

$$\le c_0\left(\|\mathcal{L}(\rho u)\|_{W^{k,p}} + \|P(\rho u)\|_{W^{k,p}(\Omega)}\right)$$

$$\le c_0\|\rho\mathcal{L}u\|_{W^{k,p}} + \underbrace{c_0\|\mathcal{L}(\rho u) - \rho\mathcal{L}u\|_{W^{k,p}} + c\|P(\rho u)\|_{W^{k,p}}}_{\le c_0(c''+c''')\|u\|_{W^{k+1,p}(K')}}$$

(15)

$$\overset{(ii)\,(iii)}{\leq} \quad c_1 \left(\|\mathcal{L}v\|_{W^{k,p}(\Omega)} + \|v\|_{W^{k+1,p}(K')} \right)$$

$$\overset{(i)}{\leq} \quad c_2 \left(\|\mathcal{L}v\|_{W^{k,p}(\Omega)} + \|v\|_{\mathcal{L}^p(\Omega)} \right) \qquad \text{qed}$$

Addenda (b = back) 03/05/2016

1. p: 11 b $\boxed{B(\langle \nabla v, g\rangle, \varphi) = \int_\Omega (\langle \nabla f, g\rangle + P_g(v))\varphi}$

We have:

$$B(\langle \nabla v, g\rangle, \varphi) - \int_\Omega \langle \nabla f, g\rangle \varphi$$

$$= \sum_{i,j,\nu} \int_\Omega \partial_i (g_\nu \partial_\nu v) a_{ij} \partial_j \varphi - \int_\Omega \sum_\nu g_\nu \partial_\nu f \varphi.$$

$$= \sum_{i,j,\nu} \int_\Omega \partial_i (g_\nu \partial_\nu v) a_{ij} \partial_j \varphi + \int_\Omega f \sum_\nu \partial_\nu (g_\nu \varphi)$$

$$\overset{\text{Assumption}}{=} \sum_{i,j,\nu} \int_\Omega \partial_i (g_\nu \partial_\nu v) a_{ij} \partial_j \varphi + \sum_{i,j,\nu} \int_\Omega \partial_i v \, a_{ij} \frac{\partial^2}{\partial x_j \partial x_\nu}(g_\nu \varphi)$$

$$= \sum_{i,j,\nu} \int_\Omega (\partial_i g_\nu) \partial_\nu v \, a_{ij} \partial_j \varphi + \cancel{\sum_{i,j,\nu} \int_\Omega g_\nu \frac{\partial^2}{\partial x_i \partial x_\nu} v \, a_{ij} \partial_j \varphi} - \sum_{i,j,\nu} \int_\Omega \frac{\partial^2}{\partial x_i \partial x_\nu} v \, a_{ij} \partial_j (g_\nu \varphi)$$

$$- \sum_{i,j,\nu} \int_\Omega \partial_i v \, \partial_\nu a_{ij} \, \partial_j (g_\nu \varphi)$$

$$\overset{\text{Cancelation}}{=} \int_\Omega \Big(\sum_{i,j,\nu} (g_\nu \partial_i (\partial_\nu a_{ij} \partial_j v) - \partial_i (a_{ij} \partial_i g_\nu \partial_\nu v) - \partial_i g_\nu a_{ij} \frac{\partial^2}{\partial x_j \partial x_\nu} \Big) \varphi$$

$$= \int_\Omega P_g(v) \cdot \varphi.$$

2. p: 14 b $\boxed{\|\rho \mathcal{L} v\|_{W^{-1,p}} \leq c \|\mathcal{L}v\|_{W^{-1,p}}}$

$$\left| \int_\Omega \varphi \rho \mathcal{L} v \right| \leq \|\mathcal{L}v\|_{W^{-1,p}} \|\nabla(\rho\varphi)\|_{\mathcal{L}^q}$$

$$\leq \|\mathcal{L}v\|_{W^{-1,p}} (\|\rho \nabla \varphi\|_{\mathcal{L}^q} + \|\varphi \nabla \rho\|_{\mathcal{L}^q})$$

$$\leq \|\mathcal{L}v\|_{W^{-1,p}} (\|\nabla \varphi\|_{\mathcal{L}^q} + \|\nabla \rho\|_{\mathcal{L}^\infty} \|\varphi\|_{\mathcal{L}^q})$$

$$\overset{\text{Poincaré}}{\leq} \|\mathcal{L}v\|_{W^{-1,p}} (1 + \|\nabla \rho\|_{\mathcal{L}^\infty} \operatorname{diam}(\Omega)) \|\nabla \varphi\|_{\mathcal{L}^q}.$$

3. p: 14 b $\boxed{\|\mathcal{L}(\rho v) - \rho(\mathcal{L}v)\|_{W^{-1,p}} \leq c \|v\|_{\mathcal{L}^p}}$

$$\left| \int_\Omega \varphi (\mathcal{L}(\rho v) - \rho \mathcal{L} v) \right| = \left| \int_\Omega (\rho \mathcal{L}^* \varphi - \mathcal{L}^*(\rho \varphi)) v \right|$$

$$\leq \|\rho(\mathcal{L}^* \varphi) - \mathcal{L}^*(\rho\varphi)\|_{\mathcal{L}^q} \|v\|_{\mathcal{L}^p}$$

$$\overset{1^{\text{st}}\text{order}}{\leq} C \|\nabla \varphi\|_{\mathcal{L}^q} \|v\|_{\mathcal{L}^p}$$

Lecture 5 09/05/2016

Chap 5: Maximum Principles

$\Omega \subset \mathbb{R}^n$ open, connected.

$a_{ij} = a_{ji} \in C^0(\Omega)$, $b_i \in C^0(\Omega)$ $i,j = 1,\dots,n$

$A(x)^T = A(x) := (a_{ij}(x))_{ji=1}^n \in \mathbb{R}^{n\times n}$ positive definite $\forall x \in \Omega$

$Lu = \sum_{i,j=1}^n a_{ij} \partial_i \partial_j u + \sum_{i=1}^n b_i \partial_i u.$ $u \in C^2(\Omega)$

Thm 1: (E. Hopf, 1927)

$$\left.\begin{array}{l} u \in C^2(\Omega), \ Lu \geq 0 \\ \text{Assume } x_0 \in \Omega \text{ s.t. } u(x_0) = \sup_\Omega u =: M \end{array}\right\} \Rightarrow u \equiv M.$$

Lemma 1: $\Omega \subset \mathbb{R}^n$ open

$A = (a_{ij})_{i,j=1}^n = A^T$ pos. definite

$u : \Omega \to \mathbb{R}$ C^2-fct'n, $x_0 \in \Omega$

$u(x_0) = \sup_\Omega u$

$\Rightarrow \sum_{i,j=1}^n a_{ij} \partial_i \partial_j u(x_0) \leq 0.$

Proof: $\exists B \in \mathbb{R}^{n\times n}$ s.t. $BB^T = A$

Define $v := B^{-1}\Omega \to \mathbb{R}$ by

$v(y) := u(By)$, $y_0 := B^{-1}x_0$

$v(y_0) = \sup_{B^{-1}\Omega} v$

$\overset{\text{Analysis I}}{\Longrightarrow} \quad \partial_i \partial_i v(y_0) \leq 0 \quad i = 1,\dots,n$

$\Longrightarrow \quad \Delta v(y_0) \leq 0$

$\parallel$ Exercise

$\sum_{i,j} a_{ij} \partial_i \partial_j v(x_0) \leq 0.$

qed

Proof of Thm 1: Assume $u \not\equiv M$

Step 1: $\exists \xi, \eta \in \Omega$, $\exists \rho > 0$ s.t.

$B_\rho(\xi) \subset \Omega$, $\eta \in \partial B_\rho(\xi)$, $u(\eta) = M$, $u < M$ on $\overline{B_\rho(\xi)} \setminus \{\eta\}$.

Proof: Choose a continuous path $\gamma : [0,1] \to \Omega$ s.t.

$\gamma(0) = x_0$, $u(\gamma(1)) < M$.

(16)

$\Rightarrow t_1 := \sup \{ t \in [0,1] \,/\, u(\gamma(t)) = M \} < 1$

$x_1 := \gamma(t_1)$

Choose $\varepsilon_1 > 0$ s.t. $\overline{B_{2\varepsilon_1}(x_1)} \subset \Omega$

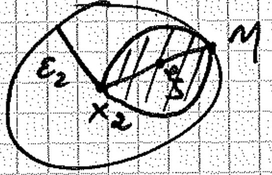
Choose $t_1 < t_2 < 1$ s.t.

$|\gamma(t_2) - x_1| < \varepsilon_1 \quad , \quad x_2 := \gamma(t_2)$

$\Rightarrow u(x_2) < M , \quad \varepsilon_2 := \sup \{ \varepsilon > 0 \mid u < M \text{ on } B_\varepsilon(x_0) \}$

$\Rightarrow \quad 0 < \varepsilon_2 \le \varepsilon_1$

$$\left. \begin{array}{l} u(x) < M \quad \forall x \in B_{\varepsilon_2}(x_2) \\ \sup\limits_{B_{\varepsilon_2}(x_2)} u = M \end{array} \right\} \Rightarrow \exists y \in \partial B_{\varepsilon_2}(x_2) \text{ s.t. } u(y) = M$$



$\xi = \frac{x_2 + y}{2}$

$\rho := \varepsilon_2 / 2$

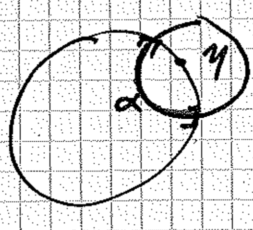
$\Rightarrow \overline{B_\rho(\xi)} \setminus \{y\} \subset B_{\varepsilon_2}(x_2).$

Step 2:

Choose $0 < r < \rho/2$ s.t. $B_r(y) \subset \Omega$

$\alpha := \partial B_r(y) \cap \overline{B_\rho(\xi)}$

$\beta := \partial B_r(y) \setminus \overline{B_\rho(\xi)}$



$\Rightarrow \begin{cases} u < M \text{ on } \alpha \\ u \le M \text{ on } \beta \end{cases}$

α cpct $\Rightarrow$

$$\boxed{\begin{array}{l} \exists \delta > 0 \text{ s.t. } u \le M - \delta \text{ on } \alpha \\ \exists \mu > 0 \quad \forall x \in \overline{B_r(y)} \quad \forall \zeta \in \mathbb{R}^n \\ \qquad \sum\limits_{i,j=1}^{n} a_{ij}(x) \zeta_i \zeta_j \ge \mu |\zeta|^2. \end{array}}$$

Step 3: $\exists$ smooth fct'n $v: \mathbb{R}^n \to \mathbb{R}$ s.t.

$$(i) \begin{cases} v > 0 & \text{on } B_\rho(\xi) \\ v = 0 & \text{on } \partial B_\rho(\xi) \\ v < 0 & \text{outside } \overline{B_\rho(\xi)} \end{cases}$$

$$(ii) \quad L v > 0 \quad \text{in } \overline{B_r(\eta)}$$

Proof: $v(x) := e^{-\theta|x-\xi|^2} - e^{-\theta\rho^2}$, $\theta > 0 \leftarrow$ satisfies (i).

$$\partial_i v = -2\theta(x_i - \xi_i)\, e^{-\theta|x-\xi|^2}$$

$$\partial_i \partial_j v = \left(4\theta^2 (x_i - \xi_i)(x_j - \xi_j) - 2\theta \delta_{ij}\right) e^{-\theta|x-\xi|^2}$$

$$\Rightarrow L v = e^{-\theta|x-\xi|^2} \Big[4\theta^2 \sum_{i,j}^{n} a_{ij}(x)(x_i - \xi_i)(x_j - \xi_j) - 2\theta \sum_{i=1} a_{ii}(x) - 2\theta \sum_{i=1}^{n} b_i(x)(x_i - \xi_i) \Big]$$

$$\geq \theta e^{\theta|x-\xi|^2} \Big(\theta \mu \underbrace{|x-\xi|^2}_{\geq (\rho/2)^2} - \text{constant} \Big) > 0$$

if $\theta > 0$ suff. large

Step 4: Proof of Thm 1:

Def'n $w := u + \varepsilon v$ with $0 < \varepsilon < \frac{\delta}{2c}$

δ as in step 2 and $c := \sup_{B_\rho(\xi)} v = 1 - e^{-\theta\rho^2} > 0$.

$$\Rightarrow w = u + \varepsilon v \leq M - \delta + \varepsilon c \leq M - \frac{\delta}{2} \quad \text{on } \alpha$$

$$w < M \quad \text{on } \beta$$

$$\Rightarrow w < M \text{ on } \partial B_r(\eta)$$

$$w(\eta) = M$$

$$L w = L u + \varepsilon L v > 0 \quad \text{on } \overline{B_r(\eta)}$$

$$\Rightarrow \exists\, x \in B_r(\eta) \text{ s.t. } w(x) = \sup_{B_r(x)} w$$

$$\Rightarrow \frac{\partial w}{\partial x_i}(x) = 0$$

$$\Rightarrow L w(x) = \sum_{i,j=1}^{n} a_{ij}(x)\, \partial_i \partial_j w(x) \overset{\text{Lemma 1}}{\leq} 0 \quad \Rightarrow \text{※}$$

Contradiction

qed

(17)

Cor 1: Ω, a_{ij}, b_i, L as in Thm 1, Ω bounded

$u \in C^2(\Omega) \cap C^0(\overline{\Omega})$

$Lu \geq 0$ in Ω

$\Rightarrow \sup_{\overline{\Omega}} u = \sup_{\partial\Omega} u$

Proof: Thm 1

Cor 2: Ω open, bounded, connected, $\partial\Omega$ smooth

$a_{ij} = a_{ji} \in C^2(\overline{\Omega})$, $b_i \in C^1(\overline{\Omega})$

L as in Thm 1. $1 < p < \infty$

$\Rightarrow L : W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \to L^p(\Omega)$ is bijective.

Proof: By Ch $\overline{\text{IV}}$ Thm 6, L is Fredholm of index 0.

Claim: L injective

Let $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ s.t. $Lu = 0$

$\overset{\text{Ch IV Thm 6}}{\Longrightarrow} \quad u \in W^{3,q}(\Omega)$ for any $q > 1$.

$\overset{k=1,\ \text{Chap II, Thm 4}}{\Longrightarrow} \quad u \in C^2(\overline{\Omega})$, $u|_{\partial\Omega} = 0$, $Lu = 0$

$\overset{\text{Cor 1}}{\Longrightarrow} \quad u \equiv 0$

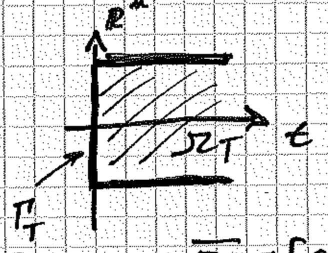
qed.

Parabolic Max. principle

$T > 0$, $\Omega \subset \mathbb{R}^n$ bounded, open

$\Omega_T := (0, T] \times \Omega$

$\Gamma_T := (\{0\} \times \Omega) \cup ([0,T] \times \partial\Omega)$



$u : \overline{\Omega_T} \to \mathbb{R}$ continuous, C^2 in Ω_T, $\overline{\Omega_T} = [0,T] \times \overline{\Omega}$.

$\boxed{\partial_t u = Lu}$, L as in Thm 1

Assume $a_{ij}, b_i \in C^0(\overline{\Omega}_T)$ $\quad i,j = 1, \dots, n$

$\sum_{i,j=1}^n a_{ij}(t,x)\, \xi_i \xi_j \geq \delta |\xi|^2 \quad \forall (t,x) \in \overline{\Omega}_T\ \forall \xi \in \mathbb{R}^n$

$$Pu = \sum_{i,j=1}^n a_{ij}\, \partial_i \partial_j u + \sum_{i=1}^n b_i\, \partial_i u - \partial_t u$$

Thm 2: $u \in C^2(\Omega_T) \cap C^0(\overline{\Omega_T})$, $Pu \geq 0 \Rightarrow \sup_{\Omega_T} u = \sup_{\Gamma_T} u$.

<u>Proof:</u> Assume $M := \sup_{\overline{\Omega}_T} u > \sup_{\Gamma_T} u$.

Choose $(t_0, x_0) \in \Omega_T$ s.t. $u(t_0, x_0) = M$

Choose

$$(1) \qquad \boxed{\theta \geq 2 \sum_{j,i=1}^{n} \|a_{ij}\|_{L^\infty} + 2 \operatorname{diam} \Omega \; \|b\|_{L^\infty(\overline{\Omega}_T, \mathbb{R}^n)}}$$

Choose $\varepsilon > 0$ s.t.

$$(2) \qquad \boxed{\varepsilon \left(e^{\theta T} - 1 \right) < M - \sup_{\Gamma_T} u}$$

Define $v(t,x) := e^{-\theta(t-t_0) - |x-x_0|^2}$

$u_\varepsilon(t,x) := u(t,x) + \varepsilon v(t,x)$

$\Rightarrow$ 1)
$$\begin{aligned} \sup_{\Gamma_T} u_\varepsilon &\leq \sup_{\Gamma_T} u + \varepsilon \sup_{\Gamma_T} v \\ &\leq \sup_{\Gamma_T} u + \varepsilon e^{\theta T} \\ &\overset{(2)}{<} M + \varepsilon = u(t_0, x_0) + \varepsilon v(t_0, x_0) \leq \sup_{\overline{\Omega}_T} u_\varepsilon . \end{aligned}$$

2)
$$\begin{aligned} P v(t,x) &= \Big[4 \sum_{i,j} a_{ij} (x_i - (x_0)_i)(x_j - (x_0)_j) - 2 \sum_{i=1}^{n} a_{ii} \\ &\quad - 2 \sum_{i=1}^{n} b_i (x_i - (x_0)_i) + \theta \Big] e^{-\theta(t-t_0) - |x-x_0|^2} \\ &> 0 \quad \text{for large } \theta . \end{aligned}$$

$\Rightarrow P u_\varepsilon(t,x) > 0 \quad \forall (t,x) \in \overline{\Omega}_T$.

Choose $(t_1, x_1) \in \overline{\Omega}_T$ s.t.

$$u_\varepsilon(t_1, x_1) = \sup_{\overline{\Omega}_T} u_\varepsilon > \sup_{\Gamma_T} u_\varepsilon$$

$x \mapsto u_\varepsilon(t_1, x)$ attains its max. in Ω.

$$P u_\varepsilon(t_1, x_1) = \underbrace{\sum_{i,j=1}^{n} a_{ij}(t,x) \frac{\partial^2 u}{\partial x_i \partial x_j}(t_1, x_1)}_{\leq 0 \text{ Lemma 1}} + \underbrace{\sum_{i=1}^{n} b_i(t_1, x_1) \frac{\partial u_\varepsilon}{\partial x_i}(t_1, x_1)}_{= 0}$$
$$\underbrace{- \frac{\partial u_\varepsilon}{\partial t}(t_1, x_1)}_{\leq 0^*} \leq 0$$

Contradiction $\Rightarrow$ ※ qed.

* $t \to u_\varepsilon(t, x_1)$ attains its maximum on $[0,T]$. If $t_1 = T$, then $\frac{\partial u_\varepsilon}{\partial t}(T, x_1) \geq 0$ (not necessarily 0).

> <u>Cor 1:</u> $Pu \leq 0$ in Ω_T, $u \in C^0(\overline{\Omega}_T) \cap C^2(\Omega_T)$
> $\Rightarrow \inf_{\overline{\Omega}_T} u = \inf_{\Gamma_T} u$

Cor 2 (Uniqueness)

$f \in C^0(\bar{\Omega}_T)$, $g \in C^0(\Gamma_T)$

$\Rightarrow \exists$ at most one sol'n $u \in C^2(\Omega_T) \cap C^0(\bar{\Omega}_T)$ of

$$(*) \begin{cases} -Pu = f & \text{in } \Omega_T \\ u|_{\Gamma_T} = g & \text{on } \Gamma_T. \end{cases}$$

Cor 3 $\quad \begin{matrix} f_1 \le f_2 \\ g_1 \le g_2 \end{matrix}$ for solutions of $(*)$ $\Rightarrow u_1 \le u_2$.

Proof: $u := u_1 - u_2$

$Pu = Pu_1 - Pu_2 = f_2 - f_1 \ge 0$

$\overset{\text{Thm 2}}{\Longrightarrow} \sup_{\bar{\Omega}_T} u = \sup_{\Gamma_T} u \le 0 \Rightarrow u_1 \le u_2$

$u|_{\Gamma_T} = g_1 - g_2 \le 0.$

qed.

Cor 4 Sol'ns of $(*)$ dep. cont'sly on g

$-Pu_1 = -Pu_2 = f$

$u_1|_{\Gamma_T} = g_1, \quad u_2|_{\Gamma_T} = g_2$

$\Rightarrow \|u_1 - u_2\|_{L^\infty(\bar{\Omega}_T)} \le \|g_1 - g_2\|_{L^\infty(\Gamma_T)}.$