The heat eq'n on $\mathbb{R}^n$

$$(*)\begin{cases}\partial_t u = \Delta u & \text{for } t>0\\ u(0,x)=u_0(x)\end{cases}$$

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$$

$u_0 : \mathbb{R}^n \to \mathbb{R}$ continuous.

Def'n: A fct'n $u : [0,\infty)\times\mathbb{R}^n \to \mathbb{R}$ is called a solution of eq'n $(*)$ if it is continuous, satisfies the initial condition $u(0,x)=u_0(x)$ for all $x\in\mathbb{R}^n$ and if $u|_{(0,\infty)\times\mathbb{R}^n}$ is smooth and satisfies $\partial_t u = \Delta u$.

Thm 3 (Existence)
For every bounded continuous fct'n $u_0 : \mathbb{R}^n \to \mathbb{R}$, there exists a sol'n of $(*)$ such that $\|u\|_{L^\infty} \le \|u_0\|_{L^\infty}$.

Fundamental Sol'n

$K : (0,\infty)\times\mathbb{R}^n \to \mathbb{R}$ given by

$$K(t,x) = K_t(x) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}, \quad t>0,\ x\in\mathbb{R}^n.$$

Lemma 2 This fct'n has the following properties
(i) $K>0$, K is smooth
(ii) $\partial_t K = \Delta K$
(iii) $K_t \in \mathcal{S}(\mathbb{R}^n)$ $\quad \forall t>0$
(iv) $\forall \delta>0 \quad \lim_{t\to 0} \left(\sup_{|x|\ge\delta} K_t(x)\right) = 0$
(v) $\forall \delta>0 \quad \lim_{t\to 0} \int_{|x|\ge\delta} K_t(x)\,dx = 0$
(vi) $\int_{\mathbb{R}^n} K_t(x)\,dx = 1. \quad \forall t>0.$

Proof (i - iv) Exercise

(v) $$\int_{|x|\ge\delta} K_t(x)\,dx = \frac{1}{(4\pi t)^{n/2}} \int_{|x|\ge\delta} e^{-\frac{|x|^2}{4t}}\,dx$$

$$y = \frac{x}{\sqrt{4t}}$$

$$= \frac{1}{\pi^{n/2}} \int_{|y| \geq \frac{\delta}{\sqrt{4t}}} e^{-|y|^2} \, dy$$

$$= \frac{\omega_{n-1}}{\pi^{n/2}} \int_{\frac{\delta}{\sqrt{4t}}}^{\infty} e^{-r^2} r^{n-1} \, dr \xrightarrow{t \to 0} 0$$

(v) $\int_{\mathbb{R}^n} K_t(x) \, dx = \frac{1}{\pi^{n/2}} \int_{\mathbb{R}^n} e^{-|y|^2} dy = 1.$

qed.

<u>Proof of Thm 3</u>

Define

$$u(t,x) := \begin{cases} \int_{\mathbb{R}^n} \overbrace{K_t(x-y) \, u_0(y) \, dy}^{(K_t * u_0)(x)} & \text{for } t > 0, x \in \mathbb{R}^n \\ u_0(x) \text{ for } t = 0, x \in \mathbb{R}^n. \end{cases}$$

<u>Continuity at $t = 0$</u>:

$$u(x) - u_0(x) = \int_{\mathbb{R}^n} K_t(x-y) \big(u_0(y) - u_0(x) \big) \, dy$$

$$\leq \underbrace{\int_{|x-y| \geq \delta} K_t(x-y) \, dy \; 2\|u_0\|_{L^\infty}}_{\xrightarrow{t \to 0} 0}$$

$$+ \int_{|x-y| \leq \delta} K_t(x-y) \, dy \sup_{|x-y| \leq \delta} |u_0(x) - u_0(y)|$$

$$\leq \varepsilon \qquad \text{for } \begin{array}{l} t > 0 \text{ suff small} \\ \delta > 0 \end{array}$$

<u>For $t > 0$</u>:

$$\partial_t u(t,x) = \int_{\mathbb{R}^n} \partial_t K_t(x-y) \, u_0(y) \, dy$$

$$\overset{(ii)}{\underset{\text{Lemma 2}}{=}} \int_{\mathbb{R}^n} \Delta K_t(x-y) \, u_0(y) \, dy$$

$$= \Delta u(t,x).$$

Also:

$$|u(t,x)| \leq \int_{\mathbb{R}^n} K_t(x-y) \, |u_0(y)| \, dy$$

$$\leq \int_{\mathbb{R}^n} K_t(x-y) \, dy \, \|u_0\|_{L^\infty} = \|u_0\|_{L^\infty}.$$

qed

(2)

Counterexample to uniqueness (Tychonoff).

$$u(t,x) : \sum_{k=0}^{\infty} g^{(k)}(t) \frac{x^{2k}}{(2k)!} \quad , t \in \mathbb{R} \ , x \in \mathbb{R} .$$

$$g(t) := \begin{cases} 0, & \text{for } t \leq 0 \\ e^{-\frac{1}{t}}, & \text{for } t > 0 \end{cases} \qquad \boxed{n = 1}$$

1) Assuming convergence

$$\partial_t u = \sum_{k=0}^{\infty} g^{(k+1)}(t) \frac{x^{2k}}{(2k)!}$$

$$= \sum_{k=1}^{\infty} g^{(k)}(t) \frac{x^{2k-2}}{(2k-2)!}$$

$$= \partial_x \partial_x u.$$

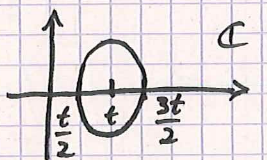
2) u continuous at $t = 0$: $\quad u(t,x) \xrightarrow{t \to 0} 0$

<u>Claim:</u> $\quad |g^{(k)}(t)| \leq \frac{2^k k!}{t^k} e^{-\frac{4}{9t^2}}$

$$\Rightarrow \ |u(t,x)| \leq e^{-\frac{4}{9t^2}} \sum_{k=0}^{\infty} \underbrace{\frac{2^k k!}{(2k)!}}_{\leq \frac{1}{k!}} \left(\frac{x^2}{t}\right)$$

$$\leq e^{-\frac{4}{9t^2} + \frac{x^2}{t}} = e^{-\frac{4 - 9t x^2}{9t^2}} \xrightarrow{t \to 0} 0$$

<u>Proof of Claim:</u>



$$g(t) = \frac{1}{2\pi i} \int \frac{e^{-\frac{1}{z^2}}}{z - t} dz$$

(Cauchy formula from Complex Analysis)

$$|g^{(k)}(t)| = \left| \frac{k!}{2\pi i} \int_{|z-t| = \frac{t}{2}} \frac{e^{-\frac{1}{z^2}} dz}{(z-t)^{k+1}} \right|$$

$$\leq \frac{k!}{2\pi} \cdot 2\pi \frac{t}{2} \frac{1}{(t/2)^{k+1}} e^{-\frac{1}{(3t/2)^2}}$$

$$= \frac{k! \, 2^k}{t^k} e^{-\frac{4}{9t^2}}$$

qed.

<u>Intuition:</u> "There is no local uniqueness! Every statement on uniqueness has to be global (behaviour at ∞). Our counter-example for example blows up at ∞".

Thm 4 $\quad v_0 : \mathbb{R}^n \to \mathbb{R}$ bnded continuous.

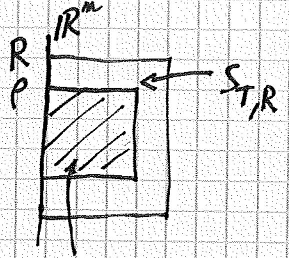
$\Rightarrow$ $\exists!$ bnded sol'n of $(*)$

i.e. if $v : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ is a bnded sol'n of $(*)$ then

$$v(t,x) = (K_t * v_0)(x) \qquad \forall t > 0 \quad \forall x \in \mathbb{R}^n.$$

Proof: Existence : Thm 3

Uniqueness : Let $v : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ be a bnded sol'n of $(*)$ with initial condition $v(0,x) = 0 \quad \forall x \in \mathbb{R}^n$.



Fix $\tau > 0$, $\rho > 0$.

(1) $\quad 0 < \alpha < \frac{1}{4\tau}$, $\varepsilon > 0$

Choose T, R s.t.

$S_{\tau,\rho} := [0,\tau] \times B_\rho$ $\quad (B_\rho \subset \mathbb{R}^n)$

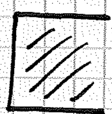
(2) $\quad \tau < T < \frac{1}{4\alpha}$, $\quad R > \rho$

(3) $\quad \varepsilon e^{\alpha R^2} > \|v\|_{L^\infty}$

Define $v_\varepsilon : S_{T,R} \to \mathbb{R}$ by

$$v_\varepsilon(t,x) := \frac{\varepsilon}{(1-4\alpha t)^{n/2}} e^{\frac{\alpha |x|^2}{1-4\alpha t}}$$

Then
$$\begin{cases} \partial_t v_\varepsilon = \Delta v_\varepsilon & \text{in } S_{R,T} \\ v_\varepsilon(t,x) \geq \varepsilon e^{\alpha R^2} > \|v\|_{L^\infty} & \text{for } |x| = R \end{cases}$$



$\Rightarrow \quad |v(t,x)| \leq v_\varepsilon(t,x) \qquad$ for $0 \leq t \leq T$, $|x| = R$ and $t = 0$, $|x| \leq R$ $\quad$ $v(0,x) = 0$

$\overset{\text{Thm 2}}{\underset{\Omega = B_R}{\Rightarrow}} \quad |v(t,x)| \leq v_\varepsilon(t,x) \qquad \forall t \in [0,T] \quad \forall |x| \leq R$

1) $v - v_\varepsilon \leq 0$ on Γ_T
$P(v - v_\varepsilon) = 0 \geq 0$
$\Rightarrow v - v_\varepsilon \leq 0$ on Ω_T

$\underset{|x| \leq \rho}{\overset{0 \leq t \leq \tau}{\Rightarrow}} \quad |v(t,x)| \leq v_\varepsilon(t,x)$

$$\leq \frac{\varepsilon}{(1-4\alpha\tau)^{n/2}} e^{\frac{\alpha\rho^2}{1-4\alpha\tau}} \xrightarrow{\varepsilon \to 0} 0.$$

2) $-v - v_\varepsilon \leq 0$ on Γ_T
$P(v - v_\varepsilon) = 0 \geq 0$
$\Rightarrow -v - v_\varepsilon \leq 0$ on Ω_T

$\Rightarrow \quad v(t,x) = 0 \qquad \forall t \in [0,\tau] \quad \forall |x| \leq \rho$ $\quad$ qed.

Thus from $-v_\varepsilon \leq v \leq v_\varepsilon$ on Γ_T, we get $-v_\varepsilon \leq v \leq v_\varepsilon$ on Ω_T $\quad$ $\overset{!}{\circ}$

Recap:

$u_0 \in C_c(\mathbb{R}^n)$.

$\overset{\text{Thm 4}}{\Longrightarrow}$ 1. $\exists!$ bnded sol'n $u : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ of

$$\begin{cases} \partial_t u = \Delta u \\ u_0(x) = u(0,x) \end{cases}$$

2. The sol'n is given by
$$u_t := u(t,\cdot) = K_t * u_0$$

3. For $1 < p < \infty$ by Young's inequality
$$\| u_t \|_{\mathcal{L}^p} = \| K_t * u_0 \|_{\mathcal{L}^p} \leq \| K_t \|_{\mathcal{L}^1} \| u_0 \|_{\mathcal{L}^p} = \| u_0 \|_{\mathcal{L}^p}$$

4. Define $S(t) : \mathcal{L}^p(\mathbb{R}^n) \to \mathcal{L}^p(\mathbb{R}^n)$ by
$$S(t)\, u_0 := K_t * u_0 \quad \text{for } u_0 \in \mathcal{L}^p(\mathbb{R}^n),\ t > 0.$$

5. <u>Exercise:</u> This is a semi-group
$$S(t+s) = S(t)\, S(s) \quad \forall t, s > 0$$
Hint: $K_{t+s} = K_t * K_s$

6. <u>Exercise:</u>
$$\lim_{t\to 0} \| S(t) u_0 - u_0 \|_{\mathcal{L}^p} = 0.$$

7. The fct'n
$$\begin{aligned} [0,\infty) &\to \mathcal{L}^p(\mathbb{R}^n) \\ t &\mapsto S(t)\, u_0 = K_t * u_0 \end{aligned}$$
is continuous for each $u_0 \in \mathcal{L}^p(\mathbb{R}^n)$

8. <u>Exercise:</u> The inf. gen. of the strongly cont. semigrp. $S(t)$ is the Laplace operator
$$\Delta : W^{2,p}(\mathbb{R}^n) \to \mathcal{L}^p(\mathbb{R}^n)$$

<u>Hint 1:</u> If $u_0 \in W^{2,p}(\mathbb{R}^n)$

$\Rightarrow$ $S(t) u_0 \in W^{2,p}(\mathbb{R}^n)$, $t > 0$
$$\Delta S(t) u_0 = S(t) \Delta u_0 = \frac{\partial}{\partial t}(S(t) u_0)$$

$\Rightarrow$ $\dfrac{S(t)u_0 - u_0}{t} = \dfrac{1}{t}\displaystyle\int_0^t S(t) \Delta u_0 \, dt \xrightarrow[t\to 0]{} \Delta u_0.$

<u>Hint 2:</u> $u_0 \in \mathcal{L}^p(\mathbb{R}^n)$, $f \in \mathcal{L}^p(\mathbb{R}^n)$
$$\lim_{t \to 0} \left\| \frac{S(t) u_0 - u_0}{t} - f \right\|_{\mathcal{L}^p} = 0$$
$\Rightarrow$ $u_0 \in W^{2,p}(\mathbb{R}^n)$, $\Delta u_0 = f$.

Check: f is weak solution of $\Delta u_0 = f$.

9. <u>Exercise:</u> $1 < p < \infty$

Assume $u : (0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ smooth

$$u_t = u(t,\cdot) \in \mathcal{L}^p(\mathbb{R}^n)$$

$$\begin{cases} (0,\infty) \to \mathcal{L}^p(\mathbb{R}^n) \\ \quad t \mapsto u(t,\cdot) \end{cases}$$

$$\lim_{t \to 0} \|u(t,\cdot)\|_{\mathcal{L}^p} = 0$$

$$\partial_t u = \Delta u \qquad \text{for } t > 0$$

$$\Rightarrow \quad u \equiv 0 \qquad (\mathcal{L}^p\text{-Uniqueness})$$

<u>Hint:</u> Fix $t > 0$

Define $\quad v(s,x) := (K_{t-s} * u_s)(x) \quad , \ 0 < s < t$

Show: $\quad \partial_s v \equiv 0$

$$v(s,\cdot) \xrightarrow{\mathcal{L}^p} u(t,\cdot) \ , \quad s \to t$$

$$\to 0 \quad , \ s \to 0$$

Lecture 7 19/05/2016

<u>Chap VI: Parabolic regularity</u>

Inhomogeneous heat eq'n:

$$u, f : \underset{t}{[0,T]} \times \underset{x}{\mathbb{R}^n} \to \mathbb{R}$$

$$(1) \boxed{\begin{cases} \partial_t u - \Delta u = f & , \ 0 \le t \le T \\ \quad u(0,x) = 0 & , \ x \in \mathbb{R}^n \end{cases}}$$

<u>Def'n:</u> $u, f \in \mathcal{L}^1([0,T] \times \mathbb{R}^n)$.

u is called <u>weak solution of (1)</u>

if
$$\left[\begin{aligned} &\int_0^T \int_{\mathbb{R}^n} u(t,x) \left(-\partial_t \varphi(t,x) - \Delta \varphi(t,x)\right) dx\, dt \\ &= \int_0^T \int_{\mathbb{R}^n} f(t,x)\, \varphi(t,x)\, dx\, dt. \end{aligned} \right.$$

for all $\varphi \in C^\infty([0,T] \times \mathbb{R}^n)$ s.t.

$\varphi(t,\cdot) \in \mathcal{S}(\mathbb{R}^n) \quad \forall t$

and $\varphi(T,x) = 0 \qquad \forall x \in \mathbb{R}^n$.

Rmk: Every strong solution s.t. u, $\partial_t u - \Delta u$ bnded is a weak sol'n.

Recall:

1) $K_t(x) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}$.

2) $S(t) : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n) : S(t)f = K_t * f$
$1 < p < \infty$. strongly cont. semigrp.

3) inf. gen. of S is $\Delta : W^{2,p}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$.

Lemma 1: $1 < p < \infty$.

$f : [0,\infty] \to L^p(\mathbb{R}^n)$ continuously differentiable.

Define

$$(3) \quad \boxed{\begin{cases} u(t,x) = \int_0^t \int_{\mathbb{R}^n} K_{t-s}(x-y)\, f(s,y)\, dy\, ds \\ \text{for } \; 0 < t \leq T, \quad u(0,x) := 0 \quad x \in \mathbb{R}^n. \end{cases}}$$

Then:

(i) $u : [0,T] \to L^p(\mathbb{R}^n)$ continuously diff.
$u : [0,T] \to W^{2,p}(\mathbb{R}^n)$ continuous
and
$\partial_t u - \Delta u = f$

(ii) f smooth, $f(t,\cdot) \in \mathcal{S}(\mathbb{R}^n)$ $\Rightarrow$ u smooth, $u(t,\cdot) \in \mathcal{S}(\mathbb{R}^n)$

Proof

Notation: $u(t) := u(t,\cdot) \in L^p(\mathbb{R}^n)$
$f(t) := f(t,\cdot) \in L^p(\mathbb{R}^n)$

$\Rightarrow u(t) := \int_0^t S(t-s) f(s)\, ds$

Apply FAI Lemma 7.1.12 $\Rightarrow$ (i).

For (ii): Differentiate in (3)

For $\alpha \in \mathbb{N}_0^n$:

$$\partial_x^\alpha u(t,x) = \int_0^t \int_{\mathbb{R}^n} K_{t-s}(x-y)\, \partial_y^\alpha f(s,y)\, dy\, ds$$

$\overset{(i)}{\Rightarrow} \partial_x^\alpha u \in C^1([0,T], L^p(\mathbb{R}^n)) \quad \forall \alpha$.

$$\partial_t v = +\Delta v + f$$
$$\partial_t^2 v = \Delta \partial_t v + \partial_t f.$$

Induction implies v is smooth

qed.

<u>Rmk:</u> We would expect that f C^0 implies still the same result. Unfortunately, this won't hold. But it will with C^0 replaced by $\mathcal{L}^q$.

<u>Lemma 2:</u>

(i) $v : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ smooth

$v(0,x) = 0 \quad \forall x \in \mathbb{R}^n$

v bounded, $\partial_t v - \Delta v$ bounded

Then:

$$\boxed{v(t,x) = \int_0^t \int_{\mathbb{R}^n} K_{t-s}(x-y)\, \underbrace{(\partial_t v - \Delta v)}_{=f}(s,y)\, dy\, ds}$$

$\forall\, 0 < t \leq T \quad \forall x \in \mathbb{R}^n$

(ii) $\varphi : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ smooth

$\varphi(T,x) = 0 \quad \forall x \in \mathbb{R}^n$

φ bounded, $\partial_t \varphi - \Delta \varphi$ bounded

Then:

$$\boxed{\varphi(s,x) = - \int_s^T \int_{\mathbb{R}^n} K_{t-s}(x-y)\, (\partial_t \varphi - \Delta \varphi)(t,y)\, dy\, dt}$$

$\forall\, 0 \leq t < T, \quad x \in \mathbb{R}^n$

<u>Proof:</u> (i) : Lemma 1 and uniqueness of bounded sol'n of heat eq'n. Ch V Thm 4.

(ii) : $v(t,x) := \varphi(T-t, x)$

qed.

<u>Lemma 3:</u> $v, f \in \mathcal{L}^1([0,T] \times \mathbb{R}^n)$

Equivalent are:

(i) v is a weak sol'n of (1).

(ii) v is given by (3).

<u>Proof:</u> <u>(ii) $\Rightarrow$ (i):</u>

$$\int_0^T \int_{\mathbb{R}^n} v(t,x)\, (\partial_t \varphi - \Delta \varphi)(t,x)\, dx\, dt.$$

$$= -\underset{t}{\int_0^T} \underset{x}{\int_{\mathbb{R}^n}} \underset{s}{\int_0^t} \underset{y}{\int_{\mathbb{R}^n}} K_{t-s}(x-y)\, f(s,y)\, (\partial_t \varphi - \Delta \varphi)(t,x) \, dy\, ds\, dx\, dt$$

$$\overset{\text{Fubini}}{=} \underset{s}{\int_0^T} \underset{y}{\int_{\mathbb{R}^n}} f(s,y) \underbrace{\left[- \underset{t}{\int_s^T} \underset{x}{\int_{\mathbb{R}^n}} K_{t-s}(y-x)\, (\partial_t \varphi - \Delta \varphi)(t,x)\, dx\, dt \right]}_{\varphi(s,y) \text{ by Lemma 2 (ii)}} dy\, ds$$

$$= \int_0^T \int_{\mathbb{R}^n} f\, \varphi$$

<u>(i) ⇒ (ii)</u> : $\tilde{v}(t,x) := \int_0^t \int_{\mathbb{R}^n} K_{t-s}(x,y)\, f(s,y)\, dy\, ds$

⇒ $v - \tilde{v}$ is a weak solution of $\partial_t (v-\tilde{v}) = \Delta (v - \tilde{v})$.

⇒ $\int_0^T \int_{\mathbb{R}^n} (v - \tilde{v}) \underbrace{(-\partial_t \varphi - \Delta \varphi)}_{\psi} = 0$ $\quad \forall \varphi$ as in the def'n.

In particular for all $\psi \in C_0([0,T] \times \mathbb{R}^n)$, we can solve $-\partial_t \varphi - \Delta \varphi = \psi$ by Lemma 1:

⇒ $v = \tilde{v}$.

qed.

<u>Some more details</u>: $\varphi(s,x) := -\int_s^T \int_{\mathbb{R}^n} K_{t-s}(x-y)\, \psi(t,y)\, dy\, dt$

$0 \le s < T$, $x \in \mathbb{R}^n$

$\overset{\text{Lemma 1}}{\Longrightarrow}$ φ is smooth, $\varphi(t,\cdot) \in \mathcal{S}(\mathbb{R}^n)$ $\forall t$

(heat eq'n in backward time) $-\partial_t \varphi - \Delta \varphi = \psi$

⇒ $\int_0^T \int_{\mathbb{R}^n} (v - \tilde{v})\, \psi = 0$.

<u>Thm 1</u> $1 < p, q < \infty$, $T > 0$,

Let $f \in L^q([0,T], L^p(\mathbb{R}^n))$.

⇒ There exists a unique solution

$v \in L^q([0,T], W^{2,p}(\mathbb{R}^n)) \cap W^{1,q}([0,T], L^p(\mathbb{R}^n))$ $\overset{*}{\subset} C^0([0,T], L^p(\mathbb{R}^n))$

of eq'n (1) i.e.

$$\boxed{\partial_t v - \Delta v = f, \quad v(0,x) = 0} \quad \forall x \in \mathbb{R}^n \quad \forall 0 < t \le T.$$

* $n=1$: $q>1$ has Sobolev embedding

Moreover, $\boxed{v(t,x) = \int_0^t \int_{\mathbb{R}^n} K_{t-s}(x-y)\, f(s,y)\, dy\, ds}$

for $0 < t \le T$, $x \in \mathbb{R}^n$.

→ So $v(0,x) = 0$ makes sense!

This result is called <u>maximal regularity</u>, because you cannot expect anything stronger.

<u>Thm 2</u> (Maximal regularity)

$\exists\, c>0 \quad \forall\, u \in C_0^\infty(\mathbb{R}, \mathbb{R}^n) \quad \forall\, T>0.$

$$(5) \qquad \left(\int_{-\infty}^{T} \| \partial_t u(t,\cdot) \|^q_{L^p(\mathbb{R}^n)} \right)^{1/q} \leq C \left(\int_{-\infty}^{T} \| \partial_t u - \Delta u \|^q_{L^p(\mathbb{R}^n)} \, dt \right)^{1/q}$$

<u>Rmk</u>: $C_0^\infty(\mathbb{R}, \mathbb{R}^n)$ is dense in $L^q(\mathbb{R}, W^{2,p}(\mathbb{R}^n)) \cap W^{1,q}(\mathbb{R}, L^p(\mathbb{R}^n))$ (Exercise)

$\Rightarrow$ (5) still holds for all u.

<u>Thm 2 $\Rightarrow$ Thm 1</u>

<u>Uniqueness</u>: (Lemma 3 for $p=1$, same argument for general p)

<u>Existence</u>: Let $f \in L^q([0,T], L^p(\mathbb{R}^n))$

Choose a sequence $f_i \in C_0^\infty((0,T) \times \mathbb{R}^n)$ s.t.

$$\lim_{i\to\infty} \| f - f_i \|_{L^q([0,T], L^p(\mathbb{R}^n))} = 0.$$

Define $u_i : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ by:

$$u_i(t,x) = \begin{cases} \int_0^t \int_{\mathbb{R}^n} K_{t-s}(x-y) f_i(s,y)\, dy = 0, & t>0,\ x\in\mathbb{R}^n \\ 0, & t \leq 0,\ x \in \mathbb{R}^n \end{cases}$$

$\overset{\text{Lemma 1}}{\Longrightarrow}$ u_i is smooth., $\partial_t u_i - \Delta u_i = f_i$

$\overset{\text{Thm 2}}{\Longrightarrow}$ $\| \partial_t u_i - \partial_t u_j \|_{L^q([0,T],L^p)} \leq c \, \| f_i - f_j \|_{L^q([0,T],L^p)}$

$$\| \Delta u_i - \Delta u_j \|_{L^q([0,T],L^p)} \overset{\text{use the eq}^n. + (\Delta).}{\leq} \| \partial_t u_i - \partial_t u_j \|_{L^q([0,T],L^p)} + \| f_i - f_j \|_{L^q([0,T],L^p)}$$

$$\leq (c+1) \, \| f_i - f_j \|_{L^q([0,T],L^p)}$$

$$\| u_i(t,\cdot) - u_j(t,\cdot) \|_{L^p(\mathbb{R}^n)} = \left\| \int_0^t K_{t-s} * (f_i - f_j)(s,\cdot)\, ds \right\|_{L^p}$$

$$\leq \int_0^t \| K_{t-s} * (f_i - f_j)(s,\cdot) \|_{L^p(\mathbb{R}^n)} \, ds$$

$$\overset{\text{Young}}{\leq} \int_0^t \underbrace{\|K_{t-s}\|_{L^1(\mathbb{R}^n)}}_{=1} \|f_i - f_j(s,\cdot)\|_{L^p(\mathbb{R}^n)}\, ds$$

$$\leq \int_0^T \|(f_i - f_j)(s,\cdot)\|_{L^p(\mathbb{R}^n)}\, ds$$

$$\overset{\text{Hölder}}{\leq} T^{\frac{q-1}{q}} \|f_i - f_j\|_{L^q([0,T],L^p)}.$$

Thus by C-Z and G.-Nirenberg interpolation,

$$\|v_i - v_j\|_{L^q([0,T],W^{2,p}(\mathbb{R}^n)) \cap W^{1,q}([0,T],L^p(\mathbb{R}^n))}$$
$$\leq C_T \|f_i - f_j\|_{L^q([0,T],L^p(\mathbb{R}^n))}.$$

$\Rightarrow$ $(v_i)_{i\in\mathbb{N}}$ is a Cauchy sequence in the B-space $L^q([0,T],W^{2,p}(\mathbb{R}^n)) \cap W^{1,q}([0,T],L^p(\mathbb{R}^n))$

Hence there is $\overset{\psi}{v}$ s.t.

$$v = \lim_{i\to\infty} v_i$$

$$\begin{array}{|l} \partial_t v_i - \Delta v_i = f_i \\ \quad \downarrow i\to\infty \\ \partial_t v - \Delta v = f. \end{array}$$

$$\begin{array}{|l} v_i(t,x) = \int_0^t \int_{\mathbb{R}^n} K_{t-s}(x-y)\, f_i(s,y)\, dy\, ds \\ \quad \downarrow i\to\infty \qquad\qquad\qquad \downarrow i\to\infty \\ v(t,x) = \int_0^t \int_{\mathbb{R}^n} K_{t-s}(x-y)\, f(s,y)\, dy\, ds \end{array}$$

qed.

<u>Proof of Thm 2 for $p=q=2$:</u>

$$\int_{\mathbb{R}^n} |\nabla v|^2 = -\int_{\mathbb{R}^n} v\,\Delta v \qquad \forall v \in C_0^\infty(\mathbb{R}^n)$$

$\Rightarrow$ For $v \in C_0^\infty(\mathbb{R}\times\mathbb{R}^n)$

$$\frac{d}{dt}\int_{\mathbb{R}^n} |\nabla v(t,x)|^2\, dx = -\frac{d}{dt}\int_{\mathbb{R}^n} v\,\Delta v$$
$$= -2\int_{\mathbb{R}^n} \partial_t v(t,x)\,\Delta v(t,x)$$

$$\Rightarrow \int_{\mathbb{R}^n} |\nabla u(T,x)|^2 dx$$

$$= -2 \int_{-\infty}^{T} \int_{\mathbb{R}^n} \partial_t u \; \Delta u$$

$$\Rightarrow \int_{-\infty}^{T} \int_{\mathbb{R}^n} |\partial_t u - \Delta u|^2$$

$$= \int_{-\infty}^{T} \int_{\mathbb{R}^n} \left(|\partial_t u|^2 + |\Delta u|^2 - 2 \partial_t u \, \Delta u \right)$$

$$= \int_{-\infty}^{T} \int_{\mathbb{R}^n} |\partial_t u|^2 + \underbrace{\int_{-\infty}^{T} \int_{\mathbb{R}^n} |\Delta u|^2 - 2 \int_{-\infty}^{T} \int_{\mathbb{R}^n} \partial_t u \, \Delta u}_{\geq 0}$$

$$\geq \int_{-\infty}^{T} \int_{\mathbb{R}^n} |\partial_t u|^2 \, dx \, dt.$$

qed