1.1. Young's Inequality. Let $1 \leq r, p, q < \infty$ such that

$$1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}.$$

Take $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$. Define the convolution $f * g$ by

$$(f * g)(x) := \int_{\mathbb{R}^n} f(y)g(x-y)\mathrm{d}y.$$

Prove that $f * g \in L^r(\mathbb{R}^n)$ and that

$$\|f * g\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}.$$

Deduce that $(L^1(\mathbb{R}^n), *)$ is a Banach algebra without unit.

Hint: Use the Hölder Inequality for three functions with $\frac{1}{r} + \frac{r-p}{rp} + \frac{r-q}{rq} = 1$ for a point-wise estimate and integrate it.

1.2. Harmonic functions on a two dimensional domain. Let $\Omega \subset \mathbb{C}$ be an open, simply connected subset of $\mathbb{C}$.

(a) Let $f : \Omega \to \mathbb{C}$ be a holomorphic function. Prove that $u := \operatorname{Re} f$ and $v := \operatorname{Im} f$ are harmonic, i.e.

$$\Delta v = \Delta u := \frac{\partial^2}{\partial x^2}u + \frac{\partial^2}{\partial x^2}v = 0.$$

(b) Let $u : \Omega \to \mathbb{R}$ be a C^2 harmonic function. Prove that there is a function $v : \Omega \to \mathbb{R}$ such that $f = u + iv : \Omega \to \mathbb{C}$ is holomorphic.

(c) Prove that if $u : \Omega \to \mathbb{R}$ is a C^2 harmonic function, then u is analytic.

(d) (Mean value property) Prove that if $u : \Omega \to \mathbb{R}$ is C^2 harmonic, then

$$u(z_0) = \int_0^1 u(z_0 + re^{2\pi it})\mathrm{d}t$$

whenever $\bar{B}_r(z_0) \subset \Omega$.

(e) (Maximum principle) Prove that if $\Omega' \subset \Omega$ is bounded, then for $u : \Omega \to \mathbb{R}$ C^2 harmonic, we have

$$\max_{\Omega'} u = \max_{\partial\Omega'} u$$

Hint: Use theorems about holomorphic functions e.g. Cauchy's theorem. For (b), consider $G := \partial_x u - i\partial_y u$ and define $v(z) := \mathrm{Im} \int_\gamma G$, where $z_0 \in \Omega$ and $\gamma : [0,1] \to \Omega$ is a smooth path such that $\gamma(0) = z_0$ and $\gamma(1) = z$.

1.3. Symmetries of PDE

(a) Prove that for $O \in \mathrm{O}(n)$ and $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ C^2 harmonic, then

$$v_O(x) := u(Ox)$$

is also harmonic where Ω is open and $x \in \Omega_O := \{x \in \mathbb{R}^n : Ox \in \Omega\}$.

(b) Prove that for $u : \Omega \subset \mathbb{R} \oplus \mathbb{R}^n \to \mathbb{R}$ a C^2 solution of the heat equation i.e.

$$\partial_t u - \Delta_x u = 0$$

where $(t, x) \in \mathbb{R} \oplus \mathbb{R}^n$ and Ω open,

$$v_{\lambda,O}(t, x) = u(\lambda^2 t, \lambda O x)$$

is also a solution of the heat equation for $\lambda > 0$, $O \in \mathrm{O}(n)$ and

$$(t, x) \in \Omega_{\lambda,O} := \{(t, x) \in \mathbb{R} \oplus \mathbb{R}^n : (\lambda^2 t, \lambda O x) \in \Omega\}.$$

(c) [1] Prove that for $u : \Omega \subset \mathbb{R} \oplus \mathbb{R}^n \to \mathbb{R}$ a C^2 solution of the heat equation, then

$$v_\epsilon(t, x) := \frac{1}{\left(\sqrt{1 + 4\epsilon t}\right)^n} \exp\left(\frac{-\epsilon \left\|x\right\|^2}{1 + 4\epsilon t}\right) u\left(\frac{t}{1 + 4\epsilon t}, \frac{x}{1 + 4\epsilon t}\right)$$

is also a solution of the heat equation for $\epsilon > 0$ and

$$(t, x) \in \Omega_\epsilon := \{(t, x) \in \mathbb{R} \oplus \mathbb{R}^n : t > -(4\epsilon)^{-1}, (\frac{t}{1 + 4\epsilon t}, \frac{x}{1 + 4\epsilon t}) \in \Omega\}.$$

Use this symmetry starting from the constant solution to get a non-trivial solution v_ϵ of the heat equation. Analyse the behaviour of v_ϵ as $t \to -(4\epsilon)^{-1}$.

1.4. Let $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ be a harmonic function and $f : \mathbb{R} \to \mathbb{R}$ a convex function[2], then $f \circ u$ is subharmonic, i.e.

$$\Delta(f \circ u) \geq 0$$

Please hand in your solutions for this sheet by Monday 29/02/2016.

[1] Thank you Yannick Krifka for pointing out a mistake in a previous version of this exercise.
[2] This means $f''(t) \geq 0$ for $t \in \mathbb{R}$.