10.1. Fact 1 about quotients Let $1 \leq p < \infty$ and $u \in W^{1,p}(\mathbb{R}^n)$. Define, for fixed $1 \leq i \leq n$ and $h \in \mathbb{R} \setminus \{0\}$ by

$$u^h(x) := \frac{u(x + he_i) - u(x)}{h}.$$

Prove that

$$\left\| u^h \right\|_{L^p(\mathbb{R}^n)} \leq \left\| \partial_i u \right\|_{L^p(\mathbb{R}^n)}.$$

Hint: Start with $u \in C^\infty(\mathbb{R}^n)$ and use the fundamental theorem of calculus.

Solution: Let $u \in C_0^\infty(\mathbb{R}^n)$. From the fundamental theorem of calculus, we know for $f_x : \mathbb{R} \to \mathbb{R} : t \mapsto u(x + te_i)$ that

$$\frac{u(x + he_i) - u(x)}{h} = \frac{1}{h} \int_0^h \frac{\mathrm{d}}{\mathrm{d}t} f_x \, \mathrm{d}t = \frac{1}{h} \int_0^h (\partial_i u)(x + te_i) \, \mathrm{d}t$$

where $x \in \mathbb{R}^n$. Now $\frac{\mathrm{d}t}{h}$ is a probability measure on $[0, h]$ and $t \to t^p$ is convex, therefore by Jensen's inequality, we have for all $x \in \mathbb{R}^n$

$$\left| u^h(x) \right|^p = \left| \frac{1}{h} \int_0^h \int_0^h (\partial_i u)(x + te_i) \, \mathrm{d}t \right|^p \leq \frac{1}{h} \int_0^h \left| \partial_i u(x + he_i) \right|^p \mathrm{d}t.$$

Hence, by Fubini's theorem, we get

$$\begin{aligned}
\int_{\mathbb{R}^n} \left| u^h(x) \right|^p \, \mathrm{d}x &\leq \int_{\mathbb{R}^n} \frac{1}{h} \int_0^h \left| \partial_i u(x + he_i) \right|^p \mathrm{d}t \, \mathrm{d}x \\
&= \frac{1}{h} \int_0^h \int_{\mathbb{R}^n} \left| \partial_i u(x + he_i) \right|^p \, \mathrm{d}x \, \mathrm{d}t \\
&= \frac{1}{h} \int_0^h \left\| \partial_i u \right\|_{L^p(\mathbb{R}^n)}^p \, \mathrm{d}t = \left\| \partial_i u \right\|_{L^p(\mathbb{R}^n)}^p
\end{aligned}$$

Now by 7.5, we have for $u \in W^{1,p}(\mathbb{R}^n)$, there is a sequence $u_j \in C_0^\infty(\mathbb{R}^n)$ such that $u_j \to u$ in $W^{1,p}(\mathbb{R}^n)$. For u_j, we have

$$\left\| u_j^h \right\|_{L^p(\mathbb{R}^n)} \leq \left\| \partial_i u_j \right\|_{L^p(\mathbb{R}^n)},$$

and, since

$$u_j^h \to u^h, \qquad \partial_i u_j \to \partial_i u \qquad \text{in } L^p(\mathbb{R}^n),$$

we can pass to the limit in theses inequalites and get

$$\left\| u^h \right\|_{L^p(\mathbb{R}^n)} \leq \left\| \partial_i u \right\|_{L^p(\mathbb{R}^n)}.$$

10.2. Fact 2 about quotients For $1 < p < \infty$, $u \in L^p(\mathbb{R}^n)$, we define u^h as in 10.1. Furthermore, we assume that

$$\sup_{h>0} \left\|u^h\right\|_{L^p(\mathbb{R}^n)} < \infty.$$

Prove that u has a weak derivative in the i-th direction in $L^p(\mathbb{R}^n)$.

Hint: Prove that $\int_{\mathbb{R}^n} \varphi u^h = \int_{\mathbb{R}^n} \varphi^{-h} u$ for all $\varphi \in C_0^\infty(\mathbb{R}^n)$ and combine this with Banach-Alaoglou.

Solution: Let us follow the hint, and establish the identity

$$\begin{aligned}\int_{\mathbb{R}^n} \varphi(x) u^h(x)\, \mathrm{d}x &= -\int_{\mathbb{R}^n} \frac{\varphi(x)}{h} u(x)\, \mathrm{d}x + \int_{\mathbb{R}^n} \frac{\varphi(x)}{h} u(x+he_i)\, \mathrm{d}x \\ &= -\int_{\mathbb{R}^n} \frac{\varphi(x)}{h} u(x)\, \mathrm{d}x + \int_{\mathbb{R}^n} \frac{\varphi(x-he_i)}{h} u(x)\, \mathrm{d}x \\ &= \int_{\mathbb{R}^n} \varphi^{-h}(x) u(x)\, \mathrm{d}x\end{aligned}$$

As all the $L^p(\mathbb{R}^n)$ under consideration are reflexive and separable, $\sup_{h>0} \left\|u^h\right\|_{L^p(\mathbb{R}^n)} < \infty$, gives us the existence of a weakly convergent subsequence $h_k \to 0$. Call the limit of this sequence $u_i \in L^p(\mathbb{R}^n)$. This give for all $\varphi \in C_0^\infty(\mathbb{R}^n)$ that

$$\int_{\mathbb{R}^n} u_i \varphi = \lim_{k\to\infty} \int_{\mathbb{R}^n} u^{h_k} \varphi = \lim_{k\to\infty} \int_{\mathbb{R}^n} u \varphi^{-h_k}.$$

Now, due to compact support, φ^{h_k} converges uniformly to $-\partial_i \varphi$, so in particular, in $L^q(\mathbb{R}^n)$ for $\frac{1}{q} + \frac{1}{p} = 1$. Thus by Hölder inequality, we have that

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} u \varphi^{-h_k} = -\int_{\mathbb{R}^n} u \partial_i \varphi.$$

This proves that u has $u_i \in L^p(\mathbb{R}^n)$ as the i-th weak derivative.

10.3. Laplace on $\mathbb{R}^n$. The purpose of this exercise is to establish the similar estimate for Δ as in the lecture course for $\Omega = \mathbb{R}^n$.[1]. The main difference is that for working on $\mathbb{R}^n$, the expression $K_j * f$ makes only sense for compactly supported functions, thus we indicate steps in this exercise to circumvent these difficulties.

We want to prove the following, for all $n \in \mathbb{N}$, $1 < p < \infty$, there is $C > 0$ such that for all $u \in C_0^\infty(\mathbb{R}^n)$, we have

$$\|\nabla u\|_{L^p(\mathbb{R}^n)} \leq \sup_{0\neq\varphi\in C_0^\infty(\mathbb{R}^n)} \frac{\int_{\mathbb{R}^n} \langle \nabla\varphi, \nabla u\rangle}{\|\nabla\varphi\|_{L^q(\mathbb{R}^n)}}. \tag{1}$$

[1] Cf. the notes provided on the webpage.

(a) Prove there is a unique bounded operator $T : L^p(\mathbb{R}^n, \mathbb{R}^n) \to L^p(\mathbb{R}^n, \mathbb{R}^n)$ such that

$$Tf = \sum_{i=1}^{n} \nabla(K_i * f_i)$$

for all $f \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^n)$. **Hint:** Use Calderòn–Zygmund.

(b) For $u \in C_0^\infty(\mathbb{R}^n)$, there exists $f \in L^p(\mathbb{R}^n, \mathbb{R}^n)$ such that for all $\varphi \in C_0^\infty(\mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^n} \langle f, \nabla\varphi \rangle = \int_{\mathbb{R}^n} \langle \nabla u, \nabla\varphi \rangle$$

where $\|f\|_{L^p(\mathbb{R}^n,\mathbb{R}^n)} = \sup_{0 \neq \varphi \in C_0^\infty(\mathbb{R}^n)} \frac{\int_{\mathbb{R}^n} |\langle \nabla u, \nabla\varphi \rangle|}{\|\nabla\varphi\|_{L^q(\mathbb{R}^n)}}$. **Hint:** Use Hahn-Banach.

(c) With f, and u as in (b), prove that $Tf = \nabla u$.

Hint:

(i) Prove that $T\nabla\varphi = \nabla\varphi$ for all $\varphi \in C_0^\infty(\mathbb{R}^n)$.

(ii) For $g \in L^p(\mathbb{R}^n, \mathbb{R}^n)$ and $h \in L^q$ with $\frac{1}{p} + \frac{1}{q} = 1$, prove that

$$\int_{\mathbb{R}^n} \langle Tg, h \rangle = \int_{\mathbb{R}^n} \langle g, Th \rangle .$$

(iii) For $g \in L^p(\mathbb{R}^n, \mathbb{R}^n)$ and $\varphi \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^n)$, prove that $\int_{\mathbb{R}^n} \langle g, \nabla\varphi \rangle = \int_{\mathbb{R}^n} \langle Tg, \nabla\varphi \rangle$.

(iv) For $g \in L^p(\mathbb{R}^n, \mathbb{R}^n)$, give a sequence $\varphi_\nu \in C_0^\infty(\mathbb{R}^n)$ with $\|Tg - \nabla\varphi_\nu\|_{L^p} \to 0$ as $\nu \to \infty$.

(v) Prove that $T^2 = T$.

(vi) Prove that for $g \in L^p(\mathbb{R}^n, \mathbb{R}^n)$ the following are equivalent.

(α) $Tg = 0$

(β) $\int_{\mathbb{R}^n} \langle g, \nabla\varphi \rangle = 0$ for all $\varphi \in C_0^\infty(\mathbb{R}^n)$.

Hint: Prove for $(\beta) \Rightarrow (\alpha)$ that $\int_{\mathbb{R}^n} \langle T^2 g, \varphi \rangle = 0$ for all $\varphi \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^n)$.

(vii) Conclude.

(d) Prove (1).

Solution:

(a) Let $T_{ji} : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ be the usual bounded operators from the Calderòn–Zygmund inequality, which on smooth functions $\varphi \in C_0^\infty(\mathbb{R}^n)$ are given by $T_{ji}(\varphi) = \partial_j(K_i * \varphi)$. Now given $f \in L^p(\mathbb{R}^n, \mathbb{R}^n)$, we can construct the operator $T : L^p(\mathbb{R}^n, \mathbb{R}^n) \to L^p(\mathbb{R}^n, \mathbb{R}^n)$ by $(Tf)_j = \sum_{i=1}^n T_{ji} f_i$. This is indeed bounded by

$$\|f\|_{L^p(\mathbb{R}^n,\mathbb{R}^n)} = \sum_{j=1}^n \left\| \sum_{i=1}^n T_{ji} f_i \right\|_{L^p(\mathbb{R}^n)} \leq \sum_{j,i=1}^n \|T_{ji} f_i\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n,\mathbb{R}^n)}$$

where

$$C = n \max_{j,i=1,\ldots,n} \|T_{ij}\|.$$

So T is a bounded operator and restricts to the right expression on $C_0^\infty(\mathbb{R}^n, \mathbb{R}^n)$.

(b) Define $Y := \{\nabla\varphi : \varphi \in C_0^\infty(\mathbb{R}^n)\} \subset L^q$ and define on Y the bounded linear functional $\Lambda : Y \to \mathbb{R}$ by

$$\Lambda(\nabla\varphi) = \int_{\mathbb{R}^n} \langle \nabla u, \nabla\varphi \rangle .$$

Then by Hahn-Banach, we can extend this functional to a bounded linear functional $\Lambda : L^q \to \mathbb{R}$ with the same operator norm

$$\|\Lambda\| = \sup_{0 \neq \varphi \in C_0^\infty(\mathbb{R}^n)} \frac{\int_{\mathbb{R}^n} \langle \nabla\varphi, \nabla u \rangle}{\|\nabla\varphi\|_{L^q(\mathbb{R}^n)}}.$$

Due to the identification, $L^p = (L^q)^*$, there is a function $f \in L^p(\mathbb{R}^n, \mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} \langle f, g \rangle = \Lambda(g)$$

for all $g \in L^q(\mathbb{R}^n, \mathbb{R}^n)$. So in particular on Y, we get

$$\int_{\mathbb{R}^n} \langle f, \nabla\varphi \rangle = \int_{\mathbb{R}^n} \langle \nabla u, \nabla\varphi \rangle$$

for all $\varphi \in C_0^\infty(\mathbb{R}^n)$. Also $\|f\|_{L^p(\mathbb{R}^n,\mathbb{R}^n)} = \|\Lambda\|$.

(c) (i) We have that $K * (\Delta\varphi) = \varphi$, so

$$(T\nabla\varphi)_j = \partial_j(\sum_{i=1}^n K_i * \partial_i\varphi) = \partial_j(K * \Delta\varphi) = \partial_j\varphi$$

where we used the fact that derivatives can be distributed freely over the factors of the convolution product.

(ii) Take $g_k, h_k \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^n)$ approximating g, h in the respective norms, then we see that it is enough by dominated convergence, to prove it for $g, h \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^n)$. We calculate

$$\begin{aligned}\int_{\mathbb{R}^n} \langle Tg, h\rangle &= \int_{\mathbb{R}^n} \left\langle \nabla(\sum_{i=1}^n K_i * g_i), h \right\rangle = -\int_{\mathbb{R}^n} \sum_{i,j=1}^n K_i * g_i \partial_j h_j \\ &= \int_{\mathbb{R}^n} \sum_{i,j=1}^n g_i K_i * \partial_j h_j = \int_{\mathbb{R}^n} \langle g, Th\rangle\end{aligned}$$

where the third equality uses $-K_j(-x) = K_j(x)$ and Fubini, the last equality uses $K_i * \partial_j h_j = \partial_i(K_j * h_j)$.

(iii) We use (i), to get

$$\int_{\mathbb{R}^n} \langle g, \nabla\varphi\rangle = \int_{\mathbb{R}^n} \langle g, T\nabla\varphi\rangle = \int_{\mathbb{R}^n} \langle Tg, \nabla\varphi\rangle$$

(iv) For $g \in L^p(\mathbb{R}^n, \mathbb{R}^n)$, we can approximate by functions of $C_0^\infty(\mathbb{R}^n)$, and so we only need to prove it for $g \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^n)$. Approximate for this function, $\sum_{i=1}^n K_i * g_i$ by smooth functions φ_ν in $W^{1,p}(\mathbb{R}^n)$, and then we get that

$$\|Tg - \nabla\varphi_\nu\| = \left\|\nabla(\sum_{i=1}^n (K_i * g_i)) - \nabla\varphi_\nu\right\|_{L^p(\mathbb{R}^n)} \leq \left\|(\sum_{i=1}^n (K_i * g_i) - \varphi_\nu\right\|_{W^{1,p}(\mathbb{R}^n)} \to 0$$

as $\nu \to \infty$.

(v) Take φ_ν as in (iii) for $g \in L^p(\mathbb{R}^n, \mathbb{R}^n)$, then from $\lim_{\nu\to\infty} \|Tg - \nabla\varphi_\nu\|_{L^p(\mathbb{R}^n,\mathbb{R}^n)} = 0$, we get by boundedness, that $\lim_{\nu\to\infty} \|T^2 g - T\nabla\varphi_\nu\|_{L^p(\mathbb{R}^n,\mathbb{R}^n)} = 0$. But by (i), we have $T\nabla\varphi_\nu = \nabla\varphi_\nu$, we get by uniqueness of limit in $L^p(\mathbb{R}^n, \mathbb{R}^n)$ that $T^2 g = Tg$. As g was arbitrary, we get $T^2 = T$.

(vi) For $(\alpha) \Rightarrow (\beta)$, we have by (iii), that $\int_{\mathbb{R}^n} \langle g, \nabla\varphi\rangle = \int_{\mathbb{R}^n} \langle Tg, \nabla\varphi\rangle = 0$. For the converse, take $\varphi \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^n)$, then there is a sequence $\psi_\nu \in C_0^\infty(\mathbb{R}^n)$, such that $\nabla\psi_\nu \to T\varphi$ in $L^q(/R^n, \mathbb{R}^n)$. Then we get that

$$\int_{\mathbb{R}^n} \left\langle T^2 g, \varphi \right\rangle = \int_{\mathbb{R}^n} \langle Tg, T\varphi\rangle = \lim_{\nu\to\infty} \int_{\mathbb{R}^n} \langle Tg, \nabla\psi_\nu\rangle = \lim_{\nu\to\infty} \int_{\mathbb{R}^n} \langle g, \nabla\psi_\nu\rangle\, 0.$$

where the second equality comes from Hölder inequality and penultimate uses (iii). As φ was arbitrary, we have by the previous point that $Tg = T^2 g = 0$.

(vii) In (b), we get for all $\varphi \in C_0^\infty(\mathbb{R}^n)$, that $\int_{\mathbb{R}^n} \langle f - \nabla u, \nabla\varphi\rangle = 0$, therefore we have $Tf = T\nabla u = \nabla u$.

(d) As $\nabla u = Tf$, we get

$$\|\nabla u\|_{L^p(\mathbb{R}^n,\mathbb{R}^n)} \leq \|T\| \, \|f\|_{L^p(\mathbb{R}^n,\mathbb{R}^n)} = \|T\| \sup_{0\neq\varphi\in C_0^\infty(\mathbb{R}^n)} \frac{\int_{\mathbb{R}^n} |\langle \nabla u, \nabla\varphi\rangle|}{\|\nabla\varphi\|}$$

where we used the property of f.

10.4. Why -1? For $1 < p < \infty$ and $\Omega \subset \mathbb{R}^n$ bounded and open, prove that the weak derivative $\partial_i u : C_0^\infty(\Omega) \to \mathbb{R}$ of a function $u \in L^p(\Omega)$ can be extended in a unique way to an element of $W^{-1,p}(\Omega)$. Thereby, we define a bounded linear operator $\partial_i : L^p(\Omega) \to W^{-1,p}(\Omega)$ and so the notation seems natural.

Solution: Indeed, we have for $\varphi \in C_0^\infty(\Omega)$ that

$$\left| -\int_\Omega u\partial_i\varphi \right| \leq \|u\|_{L^p(\mathbb{R}^n)} \|\nabla\varphi\|_{L^q(\mathbb{R}^n)} = \|u\|_{L^p(\mathbb{R}^n)} \|\varphi\|_{W_0^{1,q}(\Omega)}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Hence, $\partial_i u$ can be extended by density in a unique way to a bounded linear functional on $W_0^{1,q}(\Omega)$ of operator norm at most $\|u\|_{L^p(\mathbb{R}^n)}$, which is by definition an element of $W^{-1,p}(\Omega) = (W_0^{-1,q}(\Omega))^*$.